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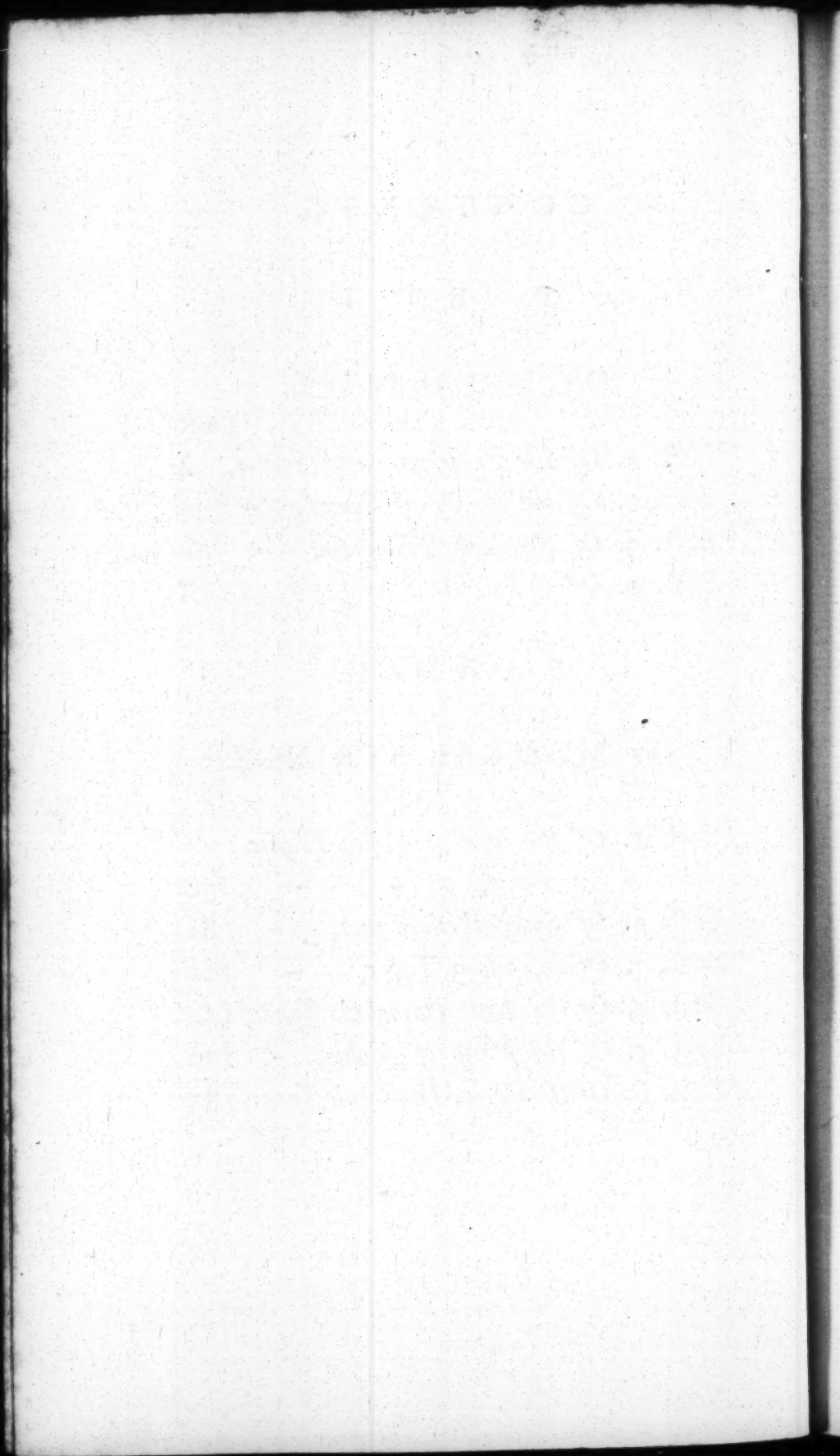
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IN the following Discourse I have endeavoured to vindicate the 47th prop. B. 2, of Newton's Principia from the objections which have been made against it; as it appears to me to be the only true principle on which the phænomena of the pulses of air can be explained. I have also endeavoured to correct such errors as, I apprehend, have been admitted into the theory of sound; and have enquired with some minuteness into the nature of the vibrations of elastic fibres, a part of natural philosophy which seems to be more fertile in consequences than has hitherto been supposed, since the ingenious writings of Hartley and Priestley on that subject. The phænomena of musical strings are also accounted for by a theory which is at least plausible: and though it is not proposed as a rigid demonstration, yet the great variety of experiments which conspire to confirm its truth, will probably be looked on as setting it far above conjecture.

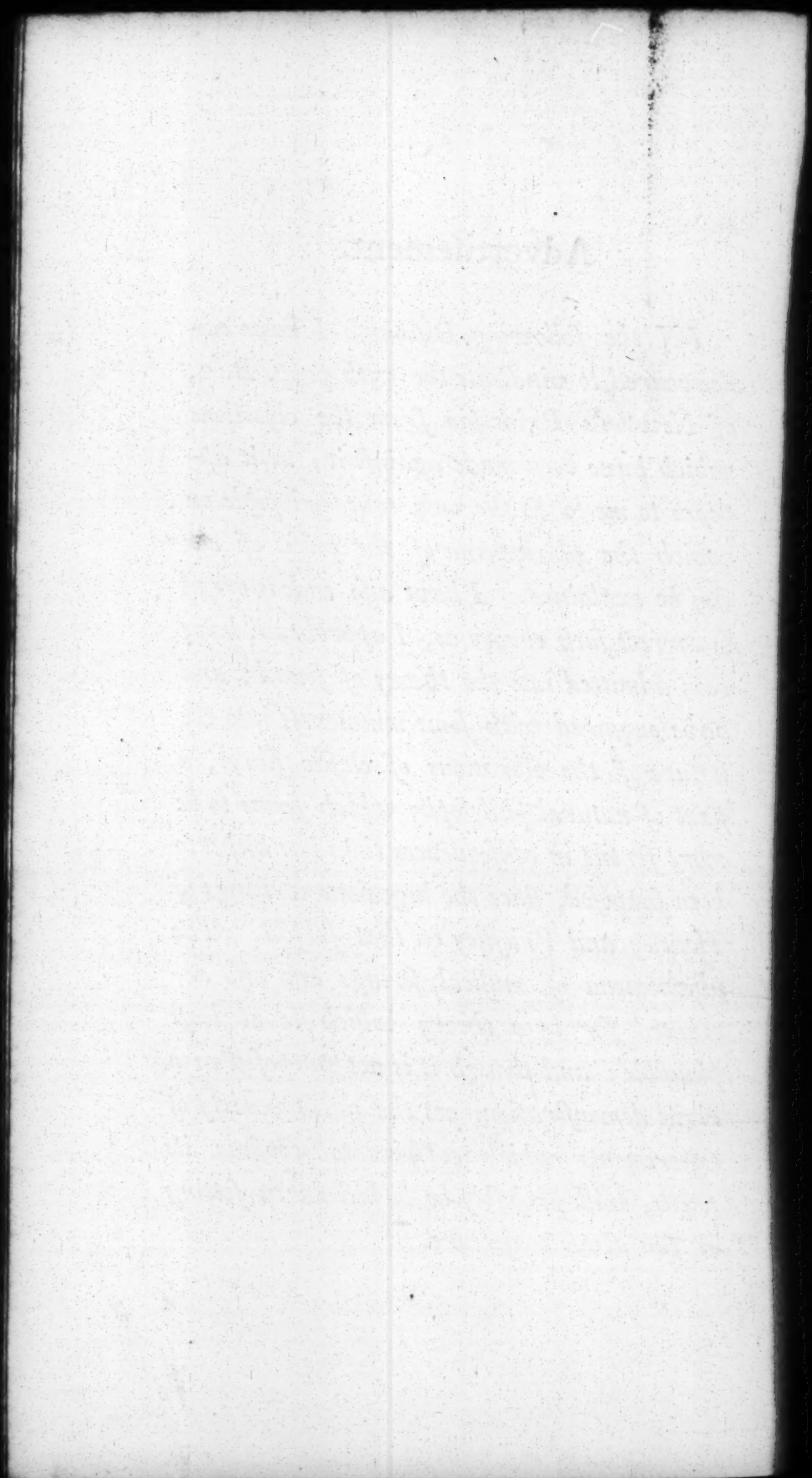


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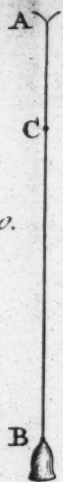


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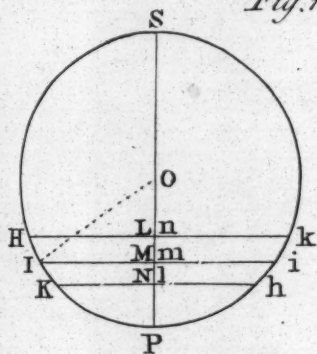


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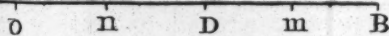


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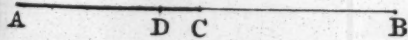


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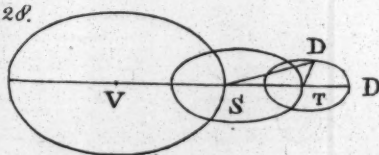


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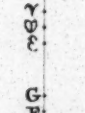
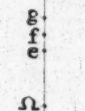
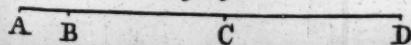


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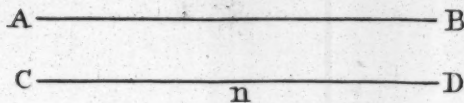


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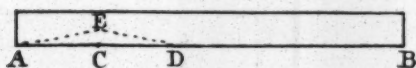


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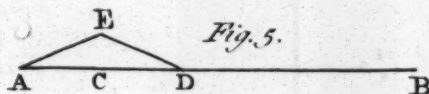


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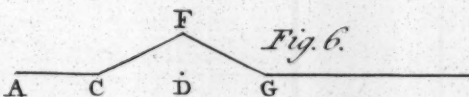


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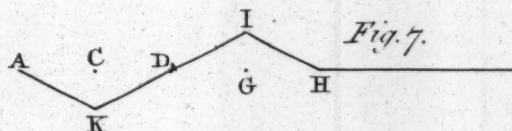


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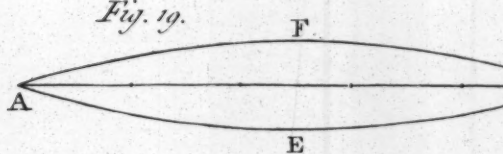


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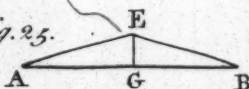


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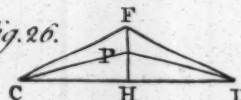
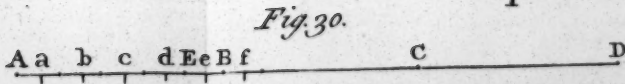
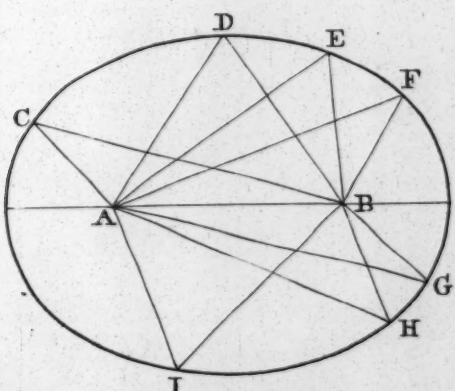
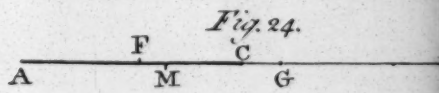
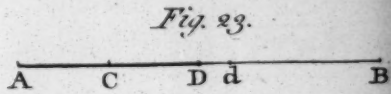
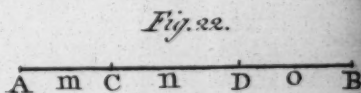
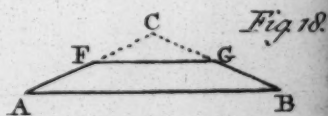
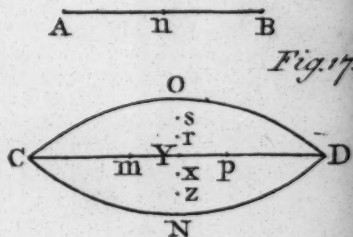
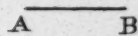
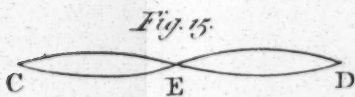
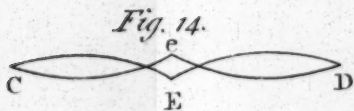
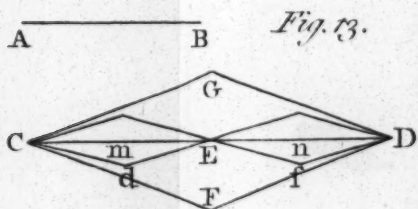
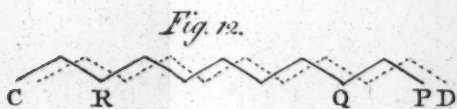
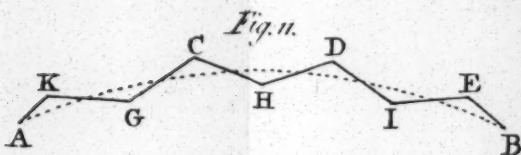
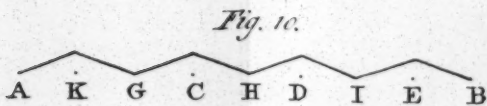
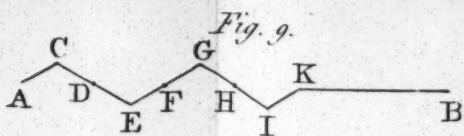
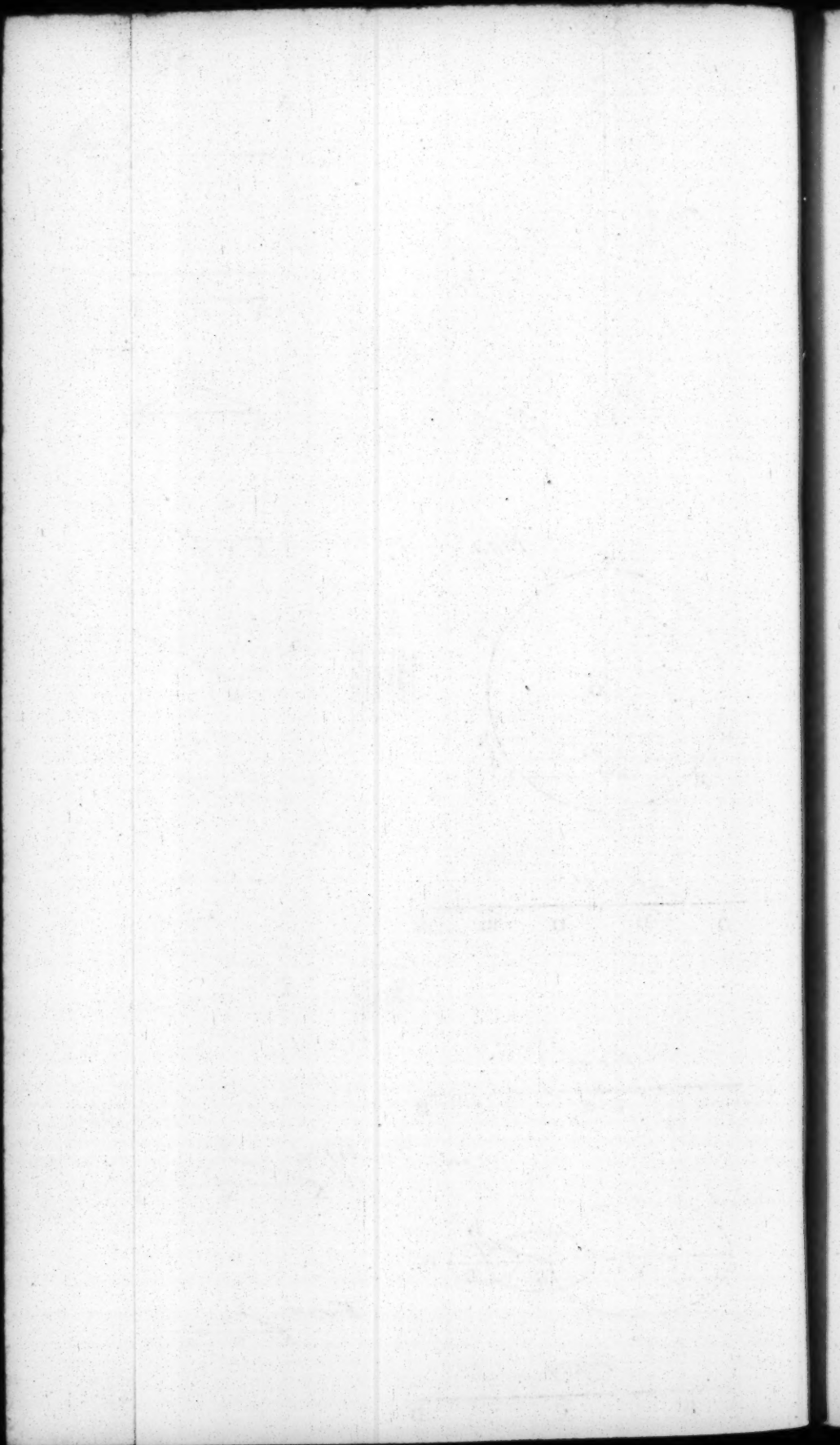


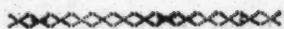
Fig. 27.







P A R T I.



O N S O U N D S.

THE phænomena of sound are so interesting and amusing, and, at the same time, so surprising, that we cannot wonder they have been found amongst the principal objects that have engaged the attention of philosophers. The ancients, however, do not appear to have made any great progress in this branch of science, considered as a subject of philosophical enquiry. They cultivated these phænomena principally with a view to the improvement of music; and, except a few vague and inconclusive theories, we scarcely find any thing considerable done till we come down to the age of the incomparable Newton. His saga-

city first explained the laws according to which the aerial pulses are propagated, and opened the way to the prosecution of this intricate subject. But, notwithstanding the attention which has been given to these discoveries, we find that they have received no accession of any consequence since his time.

The great truth which he discovered, and which is the foundation of almost all our reasoning on sound was, that "the pulses of the air are propagated in all directions from the founding body," and "that during their progress and regress, they are twice accelerated and twice retarded, according to the law of a pendulum vibrating in a cycloid:" a truth not less subtle and astonishing than his discovery, that the refracted spectrum of the sun's image is divided by the several confines of its seven colours, in the same manner as a musical

chord which produces the successive notes of an octave.

I do not propose to lay down in the following pages a regular system of the doctrine of sound, as that has been done already by a great variety of authors: I shall only endeavour to correct such errors as I conceive have been admitted into the present system, and to explain and solve such phenomena as do not appear to me to have been satisfactorily accounted for.

S E C T. I.

OF THE PROPAGATION OF SOUND.

1. **A**IR is universally admitted to be the medium of found; the experiment by which it is proved is as follows: a bell being included within the receiver of an air-pump, according as the air is in part exhausted, the found of the bell proportionally diminishes; and, when the process has been so far continued as that a perfect evacuation has been nearly effected, the found at the same time seems entirely to cease. It may be objected to the conclusiveness of this experiment, that when the air is exhausted from within, the pressure from the incumbent air on the exterior surface will prevent the found's being transmitted, in the same manner as a body is prevented from sounding by sustain-

ing a pressure on any *part* of it. But this objection may be thus obviated: let the air in the receiver be condensed instead of being exhausted, and the sound, so far from being extinguished, will be encreased; though in this case there is as great an inequality of pressure on the two surfaces as in the former.

2. In treating of the propagation of sound through this medium, the generality of writers content themselves with saying, that “ the parts of the sounding body, by going forward, compress the contiguous parts of the medium, which again compress the next, and so on; and that by returning, they suffer the compressed parts to expand; and, as these compressed parts will endeavour to expand themselves as well towards the adjacent parts of the medium which are at rest, as towards the rarer intervals of the pulses, and the quie-

fcient parts will also expand themselves into these intervals, the motion will be propagated in all directions from the founding body*." So far their rea-

* Monf. Hube, (*vide Acta Eruditorum*, an. 1762,) has endeavoured to refute this conclusion. His argument is to the following effect: "If a vertical tube, whose lower orifice is closed, be filled with water, the fides will sustain a lateral pressure; but if the lower orifice be opened, so as that the water may freely flow out, the lateral pressure will cease. In the same manner, if air be included in a vessel and compressed, its pressure will extend to all the particles in the vessel. But it does not thence follow, that the like should happen in the propagation of sound; for he that would reason so, must maintain, that the fides of a vertical tube sustain a pressure from the water which freely passes thro' it." But this example, which he has alledged, is not to the purpose. If, indeed, the particles of air included in a tube were all moved forward with the same velocity, as he supposes of the particles of water, it is evident there would be no condensation, and consequently no increase of the lateral expansion, which is all that his example tends to shew. But, if the preceding particles of water were retarded in their progress, so as that the subsequent particles should press against them, it is evident that, in such a case, there would be a lateral pressure exerted on

soning may be admitted: that is, to satisfy us that the pulses are propagated by successive goings forward and returnings backward, and successive condensations and rarefactions of the medium. But when they proceed still farther*, and pretend to draw from their arguing this immediate inference, that “the successive goings and returnings of the parts of the medium observe exactly the same law with that of the sounding body,” we must hesitate with regard to the justness of the conclusion: for it does not appear impossible, but

the sides of the tube, proportionally greater as the velocity of the subsequent exceeded that of the preceding particles. Now this is the case, and this case only, which bears any analogy to the propagation of sound; for in the aerial pulses, the preceding particles, during the first half of their progress, move with a less velocity, than those which immediately follow; a condensation therefore is produced, and consequently a lateral expansion.

* See Helsham, lect. 17, Pemberton, Martin, Rutherford, &c.

that the elastic force of the medium may be of such a nature, as to disturb and vary that motion which has been impressed on it by the sonorous body.

Mr. Rowning, indeed, has considered the constitution of the air: but the following defects appear in his reasoning; he says, that "the velocity with which each particle moves is proportional to the force which repels it, and that this force is proportional to the distance of the particle from its proper place." Now in both these assertions he seems to me to have fallen into an error; for if the diameter of a circle represent the space through which a particle vibrates, and its perimeter the time of the vibration, the force which actuates the particle will be always proportional to the cosine of the arch which represents the time of the motion; not to its distance from its proper place, which is the versed sine of that arch: and the velo-

city will not be proportional to this force, or to either the cosine or versed sine, but to the right sine of the same arch, as will appear from what Newton has demonstrated on this subject.

Into these, and other difficulties and obscurities, have the writers on sounds been betrayed, by rejecting the 47th prop. B. 2, of Newton as inconclusive reasoning; which proposition is, in fact, the only true source from whence all the valuable discoveries in the doctrine of sound are to be derived.

This demonstration is certainly obscurely stated by Newton; and by omitting a few steps which are necessary to the argument, he has given occasion to some philosophers to reject it as an argument in a circle; and which would equally tend to prove, that the pulses are propagated according to many contrary laws.

As this proposition therefore has been so severely treated, and banished out of the catalogue of demonstrations, I shall endeavour, by proposing it in as clear a light as I can, to restore it to that honour, of which, in my opinion, it has been so unjustly deprived. For this purpose I shall take the liberty of varying, in some degree, from the method of Newton.

3. The parts of all sounding bodies vibrate according to the law of a cycloidal pendulum. For they may be considered as composed of an indefinite number of elastic fibres; but these fibres vibrate according to that law. *Vide Helsham, p. 270.*

4. Sounding bodies propagate their motions on all sides *in directum*, by successive condensations and rarefactions, and successive goings forward and re-

turnings backward of the particles.
Vide prop. 43, B. 2. Newton.

5. The pulses are those parts of the air which vibrate backwards and forwards; and which, by going forward, strike (*pulsant*) against obstacles. The latitude of a pulse is the rectilineal space through which the motion of the air is propagated during one vibration of the sounding body.

6. All pulses move equally fast. This is proved by experiment; and it is found that they describe 1070 Paris feet, or 1142 London feet in a second, whether the sound be loud or low, grave or acute.

7. Prob. To determine the latitude of a pulse. Divide the space which the pulse describes in a given time (6) by the number of vibrations performed in the same time by the sounding body,

(*cor.* 1, *prop.* 24, *Smith's Harmonics*);
the quotient is the latitude.

Monf. Sauveur, by some experiments on organ-pipes, found that a body, which gives the graveſt harmonic ſound, vibrates 12 times and an half in a ſecond, and that the ſhrilleſt ſounding body vibrates 51100 times in a ſecond. At a medium, let us take the body which gives what Sauveur calls his *fixed ſound*; it performs 100 vibrations in a ſecond, and in the ſame time the pulſes deſcribe 1070 Pariſian feet; therefore the ſpace deſcribed by the pulſes whiſt the body vibrates once, that is, the latitude or interval of the pulſe, will be 10,7 feet.

8. Prob. To find the proportion which the greateſt ſpace, through which the particles of the air vibrate, bears to the radius of a circle, whoſe perimeter is equal to the latitude of the pulſe.

During the first half of the progress of the elastic fibre, or sounding body, it is continually getting nearer to the next particle; and during the latter half of its progress, that particle is getting farther from the fibre, and these portions of time are equal (*Helsham*); therefore we may conclude, that at the end of the progress of the fibre, the first particle of air will be nearly as far distant from the fibre as when it began to move: and in the same manner we may infer, that all the particles vibrate through spaces nearly equal to that run over by the fibre.

Now Monf. Sauveur (*Acad. Science, an. 1700, p. 141*) has found by experiment, that the middle point of a chord which produces his *fixed sound*, and whose diameter is $\frac{1}{2}$ th of a line, runs over in its smallest sensible vibrations $\frac{1}{18}$ th of a line, and in its greatest vibrations 72 times that space; that is, $72 \times \frac{1}{18}$ th

of a line, or 4 lines, that is, $\frac{1}{3}$ d of an inch.

The latitude of the pulses of this fixed found is 10,7 feet (7); and since the circumference of a circle is to its radius as 710 is to 113, the greatest space described by the particles will be to the radius of a circle, whose periphery is equal to the latitude of the pulse as $\frac{1}{3}$ d of an inch is to 1,7029 feet, or 20,4348 inches, that is, as 1 to 61,3044.

If the length of the string be increased or diminished in any proportion, *ceteris paribus*, the greatest space described by its middle point will vary in the same proportion. For the inflecting force is to the tending force, as the distance of the string from the middle point of vibration to half the length of the string (*see Helsham and Martin*); and therefore the inflecting and tending forces being given, the string will vi-

brate through spaces proportional to its length; but the latitude of the pulse is inverfely as the number of vibrations performed by the ftring in a given time, (7) that is, directly as the time of one vibration, or directly as the length of the ftring (*prop. 24. cor. 7. Smith's Harmonics*); therefore the greateft space thro' which the middle point of the ftring vibrates, will vary in the direct ratio of the latitude of the pulse, or of the radius of a circle whose circumference is equal to the latitude, that is, it will be to that radius as 1 to 61,3044.

9. As I have ventured, in fome particulars, to vary from the method which Newton has purfued in investigating the law of the pulses, it may not, perhaps, be unnecessary to lay before the reader a fhort view of the fteps by which I have endeavoured to illuftrate his reasoning; if, indeed, there be any

advantage in the present form, either as to perspicuity or strength.

In (10) it is proved, if the particles of air be agitated by any force whatsoever, so as successively to set out from their proper places with a motion varying according to the law of a cycloidal pendulum, that the comparative elastic forces, arising from their mutual action, by which they will be afterwards agitated, will cause them to continue that motion undisturbed to the end of their vibration. This was a necessary preliminary, for although it should be proved, that the particles of the pulses were at first successively agitated, according to the law of a pendulum, yet air might have been a medium of such a constitution, as immediately to disturb that law. From hence it follows, if we can by any means prove, that the particles of air in the pulses successively set out from their proper places,

according to the law of a pendulum, that they will finish, undisturbed, their vibrations according to the same law. Now it is the business of the two following sections, *viz.* the 11th and 12th, to demonstrate, that the particles do really set out from their proper places in this manner.

In (11) the comparative elastic force is investigated with which every particle, after it has been set in motion according to the law of a pendulum, is agitated in the different points of its progress; and thence the force with which it sets out from its proper place, before the subsequent particle has yet begun to move; which force is proved to be directly as its distance from the middle point of vibration, provided the preceding particle has made its approach to it with a motion varying according to the law already mentioned.

Now since this force, with which the particle begins to move, arises from the preceding particle's approaching it according to this law, and since the particle of an elastic fibre approaches the adjacent particle of air in this manner, it is shewn in (12) that this adjacent particle will set out with a motion varying according to the law of a pendulum. And that since this adjacent particle approaches the subsequent particle, according to the law of a pendulum, this subsequent particle must set out from its proper place in the same manner; and so on in succession. And therefore, that, since all the particles in the medium successively set out from their proper places in this manner, the motion will continue to be propagated, undisturbed, during the whole vibration. This short summary of the argument being premised, we proceed to state the demonstration itself.

10. If the particles of the aerial pulses, during any part of their vibration, be successively agitated, according to the law of a cycloidal pendulum, the comparative elastic forces arising from their mutual action, by which they will afterwards be agitated, will be such as will cause the particles to continue that motion, according to the same law, to the end of their vibration.

(*Fig. 1.*) Let AB, BC, CD, &c. denote the equal distances of the successive pulses; ABC the direction of the motion of the pulses propagated from A toward B; E, F, G three physical points of the quiescent medium, situated in the right line AC at equal distances from each other; Ee, Ff, Gg the very small equal spaces through which these particles vibrate; ϵ , ϕ , γ any intermediate places of these points.

Draw the right line PS equal to Ee, bisect it in O, and from the centre O with the radius OP describe the circle SIPb. Let the whole time of the vibration of a particle and its parts be denoted by the circumference of this circle and its proportional parts. And since the particles are supposed to be at first agitated according to the law of a cycloidal pendulum, if at any time PH, or PHSb, the perpendicular HL or bl be let fall on PS, and if Ee be taken equal to PL or Pl, the particle E shall be found in e. Thus will the particle E perform its vibrations according to the law of a cycloidal pendulum. *Prop. 52. B. 1, Principia.*

Let us suppose now, that the particles have been successively agitated, according to this law, for a certain time, by any cause whatsoever, and let us examine what will be the comparative elastic forces, arising from their mutual

action, by which they will afterwards continue to be agitated.

In the circumference $PHSb$ take the equal arches HI , IK in the same ratio to the whole circumference which the equal right lines EF , FG have to BC the whole interval of the pulses; and let fall the perpendiculars HL , IM , KN . Since the points E , F , G are successively agitated in the same manner, and perform their entire vibrations of progress and regress while the pulse is propagated from B to C , if PH be the time from the beginning of the motion of E , PI will be the time from the beginning of the motion of F , and PK the time from the beginning of the motion of G ; and therefore $E\varepsilon$, $F\varphi$, $G\gamma$ will be respectively equal to PL , PM , PN in the progress of the particles. Whence $\varepsilon\varphi$ or $EF + F\varphi - E\varepsilon$ is equal to $EF - LM$. But $\varepsilon\varphi$ is the expansion of EF in the place $\varepsilon\varphi$, and therefore this

expansion is to its mean expansion as EF — LM to EF. But LM is to IH as IM is to OP, and IH is to EF as the circumference PHS*b* is to BC; that is, as OP is to V, if V be the radius of a circle whose circumference is BC; therefore, *ex æquo*, LM is to EF as IM is to V; and therefore the expansion of EF in the place $\epsilon\phi$ is to its mean expansion, as V — IM is to V; and the elastic force existing between the physical points E and F is to the mean elastic force as $\frac{1}{V-IM}$ is to $\frac{1}{V}$ (*Gotes Pneum. Lect. 9.*) By the same argument, the elastic force existing between the physical points F and G is to the mean elastic force as $\frac{1}{V-KN}$ is to $\frac{1}{V}$; and the difference between these forces is to the mean elastic force as

$$\frac{IM-KN}{V^2-V.IM-V.KN+IM.KN} \text{ is to } \frac{1}{V}; \text{ that is, as } \frac{IM-KN}{V^2} \text{ is to } \frac{1}{V}, \text{ or as } IM-KN \text{ is to } V;$$

if only (upon account of the very narrow limits of the vibration) we suppose IM and KN to be indefinitely less than V . Wherefore, since V is given, the difference of the forces is as $IM - KN$, or as $HL - IM$ (because KH is bisected in I); that is, (because $HL - IM$ is to IH as OM is to OI or OP , and IH and OP are given quantities) as OM ; that is, if Ff be bisected in Ω as $\Omega\phi$.

In the same manner it may be shewn, that if $PHSb$ be the time from the beginning of the motion of E , $PHSi$ will be the time from the beginning of the motion of F , and $PHSk$ the time from the beginning of the motion of G ; and that the expansion of EF in the place $\varepsilon\phi$ is to its mean expansion as $EF + F\phi - E\varepsilon$, or as $EF + lm$ is to EF , or as $V + bl$ is to V in its regrefs; and its elastic force to the mean elastic force as $\frac{1}{V+bl}$ is to $\frac{1}{V}$; and that the difference of

the elastic forces existing between E and F, and between F and G is to the mean elastic force as $kn - im$ is to V; that is, directly as $\Omega\phi$.

But this difference of the elastic forces, existing between E and F, and between F and G, is the comparative elastic force by which the physical point ϕ is agitated; and therefore the comparative accelerating force, by which every physical point in the medium will continue to be agitated both in progress and regrefs, will be directly as its distance from the middle point of its vibration; and consequently, will be such as will cause the particles to continue their motion, undisturbed, according to the law of a cycloidal pendulum. *Prop. 38. l. 1. Newton.*

Newton rejects the quantity $\overline{+V \times IM + KN} + IM \times KN$ on supposition that IM and KN are indefinitely less than V.

Now although this may be a reasonable hypothesis, yet, that this quantity may be safely rejected, will, I think, appear in a more satisfactory manner from the following considerations derived from experiment: PS, in its greatest possible state, is to V as 1 is to 61,3044 (8); and therefore IM or KN, in its greatest possible state, (that is, when the vibrations of the body are as great as possible, and the particle in the middle point of its vibration) is to V as 1 is to 122,6. Hence $V^2 = 15030,76$, — $V \times \overline{IM+KN} = 245,2$ and $IM \times KN = 1$; therefore V^2 is to $V^2 - V \times \overline{IM+KN} + \overline{IM \times KN}$ as 15030,76 is to 14786,56; that is, as 61 is to 60 nearly.

Hence it appears, that the greatest possible error in the accelerating force, in the middle point is the $\frac{1}{61}$ th part of the whole. In other parts it is much less; and in the extreme points the error entirely vanishes.

We should also observe, that the ordinary sounds we hear are not produced by the greatest possible vibrations of which the sounding body is capable; and that in general IM and KN are nearly evanescent with respect to V. And very probably the disagreeable sensations we feel in very loud sounds, arise not only from IM or KN bearing a sensible proportion to V, by which means the cycloidal law of the pulses may be in some measure disturbed, but also from the very law of the motion of the sounding body itself being disturbed. For the proof of this law's being observed by an elastic fibre is founded on the hypothesis, that the space thro' which it vibrates is indefinitely little with respect to the length of the string. See *Smith's Harmonics*, p. 237, *Helsham*, p. 270.

11. If a particle of the medium be agitated, according to the law of a cy-

cloidal pendulum, the comparative elastic force, acting on the adjacent particle, from the instant in which it begins to move, will be such as will cause it to continue its motion according to the same law.

For let us suppose, that three particles of the medium had continued to move for times denoted by the arches PK, PI, PH, the comparative elastic force acting on the second during the time of its motion, would have been denoted by HL — IM, that is, would have been directly as MO (10). And if this time be diminished till I becomes coincident with P, that is, if you take the particles in that state when the second is just beginning to move, and before the third particle has yet been set in motion; then the point M will fall on P, and MO become PO; that is, the comparative elastic force of the second particle, at the instant in which

it begins to move will be to the force with which it is agitated in any other moment of time before the subsequent particle has yet been set in motion, directly as its distance from the middle point of vibration. Now this comparative elastic force, with which the second particle is agitated in the very moment in which it begins to move, arises from the preceding particle's approaching it according to the law of a pendulum; and therefore, if the preceding particle approaches it in this manner, the force by which it will be agitated, in the very moment it begins to move, will be exactly such as should take place in order to move it according to the law of a pendulum. It therefore sets out according to that law, and consequently the subsequent elastic forces, generated in every successive moment, will also continue to be of the just magnitude which should take place, in order to produce such a motion.

12. The pulses of the air are propagated from sounding bodies, according to the law of a cycloidal pendulum. (*Fig. 1.*) The point E of any elastic fibre producing a sound, may be considered as a particle of air vibrating according to the law of a pendulum (3). This point E will therefore move according to this law for a certain time, denoted by the arch IH, before the second particle begins to move; for sound is propagated in time through the successive particles of air (6). Now from that instant, the comparative elastic force which agitates F, is (11) directly as its distance from the middle point of vibration. F therefore sets out with a motion according to the law of a pendulum: and therefore the comparative elastic force by which it will be agitated until G begins to move, will continue that law (11). Consequently F will approach G in the same manner as E approached F, and the

comparative elastic force of G, from the instant in which it begins to move, will be directly as its distance from the middle point of vibration ; and so on in succession. Therefore all the particles of air in the pulses successively set out from their proper places according to the law of a pendulum, and therefore (10) will finish their entire vibrations according to the same law.

Here, by the way, we may observe the invalidity of the objection which Gabriel Cramer has advanced against this demonstration of Newton (*see the Commentary of Le Seur and Jacquier*) ; he attempts to shew, that by a mode of reasoning exactly similar, he can prove the pulses of the air to be agitated according to a law different from Newton's ; as for instance, according to the law of a heavy body ascending and descending. And the method which he pursues is as follows : on supposition that the parti-

cles were at first agitated in this manner, he proceeds to investigate what would be the comparative accelerating force of each particle in the medium during its progress and regress; and this he finds would be constant. Whence he concludes that he has, as fully as Newton, proved the truth of his proposition. But in every synthesis we are to proceed from some known established truth: this we have done in (12), it being the known law of sounding bodies, that they vibrate according to the law of a pendulum. But Cramer has not set out from any known truth; for there is no sonorous body we know of, which vibrates according to the law laid down by him.

Cor. 1. The number of pulses propagated is the same with the number of vibrations of the tremulous body, nor is it multiplied in their progress: because the little physical line $\epsilon\gamma$, (*fig. 1*) as

soon as it returns to its proper place, will there quiesce ; for its velocity, which is denoted by the sine IM, then vanishes, and its density becomes the same with that of the ambient medium. This line therefore will no longer move, unless it be again driven forwards by the impulse of the sounding body, or of the pulses propagated from it.

It has been objected to the Newtonian system, that if the first pulse of sound be driven by that which immediately follows, and that by the succeeding, and so on, it must then happen, that the more numerous the pulses, the farther will the sound be propagated ; so that a string which vibrates the longest will be heard at the greatest distance, which is contrary to known experience. (*See Encyclopædia Britann. article Acousticks*). Now this objection proceeds on the supposition,

that the particles of the pulse next the body are driven forward a certain space by its first vibration, where they would continue until driven forward again by the next; which is directly contrary to Newton's system; for he has demonstrated in this corollary, that the particles return in succession to their proper places, where their vibrations end, and where they would continue at rest, unless set in motion by a new impulse of the sounding body; while, at the same time, the condensation is uniformly carried forward through the medium, some particles being in their progress whilst others are in their regress. So that every successive vibration of the body sets in motion the very same series of particles with the preceding; which motion is propagated successively, through the subsequent particles, to the same distance, whether the vibrations of the sonorous body be repeated or not.

Cor. 2. In the extreme points of the little space through which the particle vibrates, the expansion of the air is in its natural state; for the expansion of the physical line is to its natural expansion as $V + IM$ is to V ; but IM is then equal to nothing. In the middle point of the progress the condensation is greatest; for IM is then greatest, and consequently the expansion $V - IM$ least. In the middle of the regress, the rarefaction is greatest; for im , and consequently $V + im$ is then greatest.

13. To find the velocity of the pulses, the density and elastic force of the medium being given.

This is the 49th prop. B. 2. Newton, in which he shews, that whilst a pendulum, whose length is equal to the height of the homogeneous atmosphere, vibrates once forwards and backwards, the pulses will describe a space equal to

the periphery of a circle described with that altitude as its radius.

Cor. 1. He thence shews, that the velocity of the pulses is equal to that which a heavy body would acquire in falling down half the altitude of that homogeneous atmosphere; and therefore, that all pulses move equally fast, whatever be the magnitude of PS, or the time of its being described; that is, whether the tone be loud or low, grave or acute. *See Hales de Sonis, § 49.*

Cor. 2. And also, that the velocity of the pulses is in a ratio compounded of the direct subduplicate ratio of the elastic force of the medium, and the inverse subduplicate of its density. Hence sounds move somewhat faster in summer than in winter. *See Hales de Sonis, p. 141.*

14. The strength of a tone is as the moment of the particles of air. The

moment of these particles, (the medium being given) is as their velocity: and the velocity of these particles is as the velocity of the string which sets them in motion (12). The velocities of two different strings are equal when the spaces which they describe in their vibrations are to each other as the times of these vibrations: therefore, two different tones are of equal strength, when the spaces, through which the strings producing them vibrate, are directly as the times of their vibration.

15. Let the strength of the tones of the two strings AB, CD, which differ in tension only (*fig.* 25, 26,) be equal. Quere the ratio of the inflecting forces F and f . From the hypothesis of the equality of the strength of the tones it follows (14), that the space GE must be to the space HF as $f^{\frac{1}{2}}$ to $F^{\frac{1}{2}}$, (*Smith's Harm. Prop.* 24. *Cor.* 4.) Now the forces inflecting AB, CD through the equal spaces GE, HP are to each other as the

tending forces, that is, as F to f , (*Malcolm's Treatise on Music*, p. 52.) But the force inflecting CD through HP is to the force inflecting it through HF as HP or GE to HF , (*ib.* p. 47.) that is, by the hyp. as $f^{\frac{1}{2}}$ to $F^{\frac{1}{2}}$. Therefore *ex æquo*, the forces inflecting AB and CD , when the tones are equally strong, are to each other as $F \times f^{\frac{1}{2}}$ to $f \times F^{\frac{1}{2}}$, or as $F^{\frac{1}{2}}$ to $f^{\frac{1}{2}}$. That is, the forces necessary to produce tones of equal strength in various strings which differ only in tension, are to each other in the subduplicate ratio of the tending forces, that is, inversely as the time of one vibration, or directly as the number of vibrations performed in a given time. Thus, if CD be the acute octave to AB , its tending force will be quadruple that of AB , (*Malcolm's Treatise on Music*, p. 53;) and therefore to produce tones of equal strength in these strings, the force impelling CD must be double that impelling AB ; and so in other cases.

Suppose now that the strings AB, CD (*fig.* 26, 27,) differ in length only. The force inflecting AB through GE is to the tending force, which is given, as GE to AG; and this tending force is to the force inflecting CD through the space HP equal to GE, as HD to HP. Therefore, *ex æquo*, the forces inflecting AB and CD through the equal spaces GE and HP are to each other as HD to AG, or as CD to AB. But the force inflecting CD through HP is to the force inflecting it through HF, as HP or GE to HF, that is, because these spaces are as the times (14), as AB to CD. Therefore, *ex æquo*, the forces inflecting AB and CD, when the tones are equally strong, are to each other in a ratio of equality. Hence we should suppose, that in this case, an equal number of equal impulses would generate equally powerful tones in these strings. But we are to observe, that the longer the string the greater, *ceteris paribus*, is the space through which a

given force inflects it (*Malcolm*); and therefore whatever diminution is produced in the spaces through which the strings move in their successive vibrations, arising either from the want of perfect elasticity in the strings, or from the resistance of the air, this diminution will bear a greater proportion to the less space, through which the shorter string vibrates. And this is confirmed by experience; for we find that the duration of the tone and motion of the whole string exceeds that of any of its subordinate parts. Therefore, after a given interval of time, a greater quantity of motion will remain in the longer string; and consequently after the successive equal impulses have been made, a greater degree of motion will still subsist in it. That is, a given number of equal impulses being made on various strings differing in length only, a stronger sound will be produced in that which is the longer.

S E C T. II.

OF THE DECAY OF SOUND.

16. **I**N accounting for the decay of sound, according to the distance from the founding body, we are told, that whatever be the force wherewith the founding body acts on the first spherical shell of air, with the very same force does that shell act on the second, and that again upon the third, and so on; so that the force condensing the air in the several shells is given; consequently, the condensations which it produces in these shells must be inversely as the resistance it meets with: but the resistances are as the shells, and therefore, since these increase continually in the same proportion with the squares of the

distances from the centre, their condensations must decrease in the same manner. And hence it is, say the writers on these subjects, that sounds grow less and less audible the farther they go from the sounding body. *See Merfennus, Helsham, &c.*

But this reasoning is founded on the hypothesis, that a body in motion communicates to another, which it impells, a quantity of motion equal to its own. This however is not always true; and it is, particularly in this case, contradicted by their own theory; for as these shells are elastic, and therefore as motion is communicated from a smaller elastic body to a greater, there should be an increase of motion; and consequently, the compressing forces in the several shells are not of the same magnitude. And further, the very same mode of arguing may be applied to overturn their theory of the increase of sound in speaking-trum-

pets; where, in the same manner, we may say, that the force with which the first cylinder of air acts on the second, is equal to that with which the second acts on the third, and so on: therefore the force, with which the several cylinders of air act on each other, is given, and consequently the condensations will be inversely as the resistances: but these resistances, in the best contrived tubes, increase in geometrical progression; therefore the condensations, and consequently the strength of the sound, should decrease in the same proportion.

But even supposing that we are to consider the motion of sound as communicated in this manner from one elastic shell of air to another, we shall find, that the decay of sound would not follow the law which has been laid down, *viz.* the inverse duplicate of the distance. For let a series of elastic bodies be represented by A, B, C, &c. motion be-

ing communicated from A to B, and from B to C, and so on. Let the velocities of those bodies be denoted by a, b, c , &c. the general series representing the quantities of motion of each body in succession, according to the law of collisions, will be Aa .

$\frac{2AaB}{A+B} = Bb \cdot \frac{2BbC}{B+C} = Cc$, &c. (*Helfham*, p. 361), where we see, that the quantity of motion in every preceding shell is less than that of the succeeding, in the ratio of the sum of the two shells to twice the greater; and the series exhibiting the velocities will be,

$a \cdot \frac{2Aa}{A+B} = b \cdot \frac{2Bb}{B+C} = c$, &c. whence we see, that the velocity of every preceding shell is greater than that of the succeeding, in the proportion of the sum of the two shells to twice the less. Therefore in this case, since the radii of the shells increase as the natural numbers 1, 2, 3, 4, &c. the shells themselves will be as 1, 4, 9, 16,

&c. and the velocities, according to the above series, will be had by multiplying into each other the terms of the following series, $1, \frac{2}{3}, \frac{8}{27}, \frac{1}{25},$ &c. therefore the velocity, and consequently the force of the sound, decreases through these shells in the following proportion,

$1 \cdot \frac{2}{3} \cdot \frac{16}{27} \cdot \frac{288}{16225},$ &c. that is, as

$1 \cdot \frac{1}{2\frac{1}{2}} \cdot \frac{1}{4\frac{1}{2}} \cdot \frac{1}{5\frac{185}{288}},$ &c. whereas agreeable to the received theory, it should decrease in the following proportion, $1,$

$\frac{1}{4}, \frac{1}{9}, \frac{1}{16},$ &c.

It might be expected that I should not here pass over, unobserved, the experiment adduced by Sir Isaac Newton in favour of the received doctrine. Let there be two sonorous bodies, says he, whose tones are equally strong, the one in the open air, the other included in a receiver partly exhausted. If their distances be in a subduplicate ratio of the densities of the air in which they vibrate, their tones will strike the ear

with equal force. For instance, if only one hundredth part of the whole air remain in the receiver, the sound should be one hundred times more languid; and therefore should be heard no more distinctly, than if a person in the open air should recede ten times the distance from the sounding body. Now it appears, that the sounds in both cases are equally strong. Therefore, the sounds must have decreased in the inverse duplicate ratio of the distance. (*See Principia, B. 2. prop. 47. first edition; or Hales de Sonis, p. 41*). This experiment is, for several reasons, inconclusive; it is next to impossible to contrive it so as that the original sounds shall be exactly in the direct ratio of the densities of the air; and even admitting this were possible, yet, no musician could pretend to determine, whether two sounds which approach nearly to an equal degree of strength are, in fact, accurately so or not. But in truth

Newton himself seems to have been conscious of the imperfection of this experiment, from his omitting it in the subsequent editions of his Principia.

The principal cause of the decay of sound seems to me to be the want of perfect elasticity in the air, whence it arises, that every subsequent particle has not the entire motion of the preceding particle communicated to it, as in the case of equal and perfectly elastic bodies; whence the farther the motion is propagated, the more will the velocity with which the particles move be diminished, and therefore the less the space PS (*fig. 1*), thro' which they perform their vibrations; and the less the *sinus totus* IM. But in all vibrations of the sounding body, the latitude of the pulses, and consequently the quantity V, remains the same; therefore the farther the motion is propagated the less will be the quantity $\frac{I}{V-IM}$ which

expresſes the condenſation; therefore the farther the pulse is propagated, the more is the denſity, and confequently the impulse on the drum of the ear, diminished.

And that the defect of perfect elaſticity in the air is a principal cauſe of the decay of ſound appears from this, that ſounds are perceived more diſtinctly when north, north-eaſt, and eaſterly winds prevail, as Derham and Kircher have obſerved, (*ſee Phil. Tranſ. an. 1705*); at which time the air is dry, (*ſee Du Luc, V. 2. p. 203*), and confequently more elaſtic, for vapours diffuſed through the atmosphere, unleſs dilated by intense heat, diminiſh the ſpring of the air, (*ibid. p. 194*). And therefore the ſpaces deſcribed by the vibrating particles do not decreaſe ſo rapidly, though the barometer be at that time higheſt, (*ſee Du Luc, V. 2. p. 203*), and confequently the maſs of air, to be moved, greateſt.

We may also deduce the same inference from observing, that the tone of a sounding body, at a given distance, gradually languishes; which arises from this, that in every successive vibration, the space through which the string, and consequently every particle of air, vibrates gradually diminishes. And therefore we may attribute the decay of sound at different distances to the same cause; that is, to a diminution of the space through which the particles vibrate, arising, in a great measure, from their want of perfect elasticity.

S E C T. III.

OF SPEAKING-TRUMPETS.

17. **W**RITERS in general account for the augmentation of sound in speaking-trumpets by the communication of motion from a smaller elastic body to a greater; in which case there is always an increase of motion, as appears from (16). And therefore, as this happens in the successive plates of air through the whole length of the tube, the last plate must emerge from the mouth of the tube with a very considerable increase of motion.

From this reasoning they have been led to consider the best form for such

tubes, and have concluded that to be the best, which is generated by the revolution of the logarithmic curve round its axis, as in a tube of this form the elastic bodies will increase in such a manner as most to increase the quantity of motion. (*See Helsham, p. 75, &c.*) But though there may be, in these cases, an increase of motion, there certainly is a decrease of velocity (16), therefore the space PS (p. 46), and consequently IM will diminish (*fig. 1*); therefore the condensation cannot possibly derive any augmentation from the increase of the whole motion merely. Though the last plate of air emerges from the mouth of the tube with a considerable increase of motion, yet, as its velocity will have decreased, the density, and therefore the force of the pulse, will have decreased also. The ear in all cases is affected by a cylinder of air of a given base; and therefore in estimating the magnitude of the stroke, we

are not to consider the quantity of motion which may subsist in the whole body of the atmosphere, but what subsists in that part of it only which strikes on the tympanum. And therefore, though the motion of the whole mass may be increased, yet, the motion of that particular cylinder which produces the impulse on the ear, may be, and really is, diminished (16); even according to their own theory.

18. Martin, V. 2. p. 247, and many others, (*see Encyclopædia Brittan. article Acousticks,*) derive the augmentation of the sound principally from its confinement; and calculate its increase in the following manner: the intensity of the naked voice is to that with which it issues from the mouth of the tube, as the segment of the spherical surface, described with the length of the tube as its radius, and whose base is the orifice of the tube, to that entire spheri-

cal surface. Because, the voice, by receding in every direction from the mouth, is dilated through that entire surface, but at its issuing from the tube, it is all confined to the above-mentioned segment; and the intensities of the voice will be reciprocally as the masses of air to be moved. But it is evident, that the confinement of the voice can have little effect in increasing the strength of the sound, as this strength depends on the velocity with which the particles move. Were this reasoning conclusive, the voice should issue through the smallest possible orifice; cylindrical tubes would be preferable to any that increased in diameter; and the less the diameter, the greater would be the effect of the instrument, because the plate or mass of air to be moved would, in that case, be less, and consequently the effect of the voice the greater; all which is contradicted by experience.

19. The cause of the increase of sound in these tubes must therefore be derived from some other principles: and amongst these we shall probably find, that what the ingenious Kircher has suggested in his *Phonurgia* is the most deserving of our attention. He tells us, that "the augmentation of the sound depends on its reflection from the tremulous sides of the tube; which reflections, conspiring in propagating the pulses in the same direction, must increase its intensity." Newton also seems to have considered this as the principal cause, in the scholium of prop. 50. B. 1. Princip. when he says, "we hence see why sounds are so much increased in stentorophonic tubes, for every reciprocal motion is, in each return, increased by the generating cause."

Farther, when we speak in the open air, the effect on the tympanum of a

distant auditor is produced merely by a single pulse. But when we use a tube, all the pulses propagated from the mouth, except those in the direction of the axis, strike against the sides of the tube, and every point of impulse becoming a new centre, from whence the pulses are propagated in all directions, a pulse will arrive at the ear from each of those points; thus, by the use of a tube, a greater number of pulses are propagated to the ear, and consequently the sound increased. The confinement too of the voice may have some effect tho' not such as is ascribed to it; for the condensed pulses produced by the naked voice, freely expand every way; but in tubes, the lateral expansion being diminished, the direct expansion will be increased, and consequently the velocity of the particles, and the intensity of the sound. The substance also of the tube has its effect; for it is found by experiment, that the more elastic

the substance of the tube, and the more porous, and consequently the more susceptible it is of these tremulous motions, the stronger is the sound. Of different rooms of the same form and size, we find that some deaden the voice of the speaker, whilst in others we perceive it distinct and forcible. This can only arise from the different degrees of elasticity, and consequently of vibration in the walls of these chambers, by which the pulses of the air are thus variously affected.

If the tube be laid on any non-elastic substance, it deadens the sound, because it prevents the vibratory motion of the parts. The sound is increased in speaking-trumpets, if the tube be suspended in the air; because the agitations are then carried on without interruption. These tubes should increase in diameter from the mouth-piece, because the parts, vibrating in directions

perpendicular to the surface, will conspire in impelling forward the particles of air, and consequently, by increasing their velocity, will increase the intensity of the sound: and the surface also increasing, the number of points of impulse and of new propagations will increase proportionally. The several causes, therefore, of the increase of sound in these tubes seem to be, the diminution of the lateral, and consequently the increase of the direct expansion and velocity of the included air. Secondly, the increase of the number of pulses, by increasing the points of new propagation. And thirdly, the reflections of the pulses from the tremulous sides of the tube, which impel the particles of air forward, and thus increase their velocity.

Sounds in passing over a rough surface are not varied as to *velocity*, for

that depends merely on the state of the air as to density and elasticity (13). But they decay in *strength*, because the vibrations of the surface, being in various directions, will counteract the motion of the sound. In distinguishing these effects we should observe, that the strength of the tone does not depend on the greater velocity of the *pulses*, but of the *particles* of air which constitute them.

S E C T. IV.

O F E C H O S.

20. **T**HE phænomena of echos are thus accounted for by the writers on acousticks: If sounds in their progress strike against a proper obstacle, they will partly divaricate into the spaces behind the obstacle; and partly rebounding from thence, as from a centre, they will be again propagated back in an angle equal to the angle of incidence, and by this reflection will generate a second sound, or echo. (*See Kircher's Phonurgia; Jacquier and Le Seur on Newton, B. 2. § 324; Martin's Phil. Brit. V. 2. p. 237; Encyclop. Brit. article Acousticks*). Writers have, in general, reasoned on the pulses of air as on the

rays of light, which are reflected in an angle equal to the angle of incidence. "As to the reflection of sound," says M. Lambert, "we know that it observes the law of all other reflections, that is to say, that the angle of reflection is equal to the angle of incidence; on which is founded the method of constructing artificial echos." See *Berlin Transactions*, vol. 19. p. 91.

But this is, no doubt, an erroneous position, which has probably taken its rise from carrying the analogy between light and sound too far. For in sound, every point of impulse becomes a centre from which a new series of pulses are propagated in every angle whatsoever. The principles by which these phænomena may be explained are few and simple, viz. that every point, against which the pulses strike, becomes a centre of a new series of pulses, (*prop. 43. B. 2. Newton*);

and secondly, that sound describes equal spaces in equal times (6 and 13). Therefore, when any sound is propagated from a centre, and its pulses strike against a variety of obstacles, if the sum of the right lines drawn from that point to each of the obstacles, and from each obstacle to a second point, be equal; then will the latter be a point in which an echo will be heard, (*fig. 3*). Thus, let A be the point from which the sound is propagated in all directions, and let the pulses strike against the obstacles C, D, E, F, G, H, I, &c. each of these points becomes a new centre of pulses by the first principle, and therefore from each of them one series of pulses will pass through the point B. Now if the several sums of the right lines $\overline{AC+CB}$, $\overline{AD+DB}$, $\overline{AE+EB}$, $\overline{AG+GB}$, $\overline{AH+HB}$, $\overline{AI+IB}$, &c. be all equal to each other, it is obvious that the pulses propagated from A to these points, and again from these points to B, will all

arrive at B at the same instant, according to the second principle; and therefore, if the hearer be in that point, his ear will at the same instant be struck by all these pulses. Now it appears from experiment, (*see Musschenbroek, V. 2. p. 210*), that the ear of an exercised musician can only distinguish such sounds as follow one another at the rate of 9 or 10 in a second, or any slower rate: and therefore for a distinct perception of the direct and reflected sound, there should intervene the interval of $\frac{1}{9}$ th of a second; but in this time sound describes 1142 or 127 feet nearly (6). And therefore, unless the sum of the lines drawn from each of the obstacles to the points A and B exceeds the interval AB by 127 feet, no echo will be heard at B. Since the several sums of the lines drawn from the obstacles to the points A and B are of the same magnitude, it appears that the curve passing through all the points

C, D, E, F, G, H, I, &c. will be an ellipse, (*prop.* 14. *B.* 2. *Ham. Con.*) Hence all the points of the obstacles which produce an echo, must lie in the surface of the oblong sphæroid, generated by the revolution of this ellipse round its major axis.

As there may be several such sphæroids of different magnitudes, it follows, that there may be several different echos of the same original sound. And as there may happen to be a greater number of reflecting points in the surface of an exterior sphæroid than in that of an interior, a second or a third echo may be much more powerful than the first, provided that the superior number of reflecting points, that is, the superior number of reflected pulses propagated to the ear, be more than sufficient to compensate for the decay of sound which arises from its being propagated through a

greater space. This is finely illustrated in the celebrated echos at the lake of Killarney in Kerry, where the first return of the sound is much inferior in strength to those which immediately succeed it.

It is not absolutely necessary that the reflecting points C, D, E, F, &c. producing an echo, should lie accurately in the periphery of an ellipse; for if the sums of the right lines $\overline{AC+CB}$, $\overline{AD+DB}$, &c. do not differ from each other by more than 127 feet, the pulses propagated from these points will not be distinguishable. However the nearer these sums approach to equality, the more accurate will be the coincidence of the pulses, and consequently the echo will be more distinct.

21. From what has been laid down it appears, that for the most powerful echo, the sounding body should be in

one focus of the ellipse which is the section of the echoing sphæroid, and the hearer in the other. However an echo may be heard in other situations, though not so favourably; as such a number of reflected pulses may arrive at the same time at the ear as may be sufficient to excite a distinct perception. Thus a person often hears the echo of his own voice; but for this purpose he should stand at least 63 or 64 feet from the reflecting obstacle, according to what has been said before. At the common rate of speaking, we pronounce not above three syllables and an half, that is, seven half syllables in a second; therefore, that the echo may return just as soon as three syllables are expressed, twice the distance of the speaker from the reflecting object must be equal to 1000 feet; for, as sound describes 1142 feet in a second, $\frac{6}{7}$ ths of that space, that is, 1000 feet nearly, will be described while six half, or three

whole syllables are pronounced: that is, the speaker must stand near 500 feet from the obstacle. And in general, the distance of the speaker from the echoing surface, for any number of syllables, must be equal to the seventh part of the product of 1142 feet multiplied by that number.

In churches we never hear a distinct echo of the voice, but a confused sound when the speaker utters his words too rapidly; because the greatest difference of distance between the direct and reflected courses of such a number of pulses as would produce a distinct sound, is never in any church equal to 127 feet, the limit of echos.

But though the first reflected pulses may produce no echo, both on account of their being too few in number, and too rapid in their return to the ear; yet it is evident, that the reflecting surface

may be so formed, as that the pulses which come to the ear after two reflections or more may, after having described 127 feet or more, arrive at the ear in sufficient numbers, and also so nearly at the same instant, as to produce an echo, though the distance of the reflecting surface from the ear be less than the limit of echos. This is confirmed by a singular echo in a grotto on the banks of the little brook called the Dinan, about two miles from Castlecomber, in the county of Kilkenny. As you enter the cave, and continue speaking loud, no return of the voice is perceived ; but on your arriving at a certain point, which is not above 14 or 15 feet from the reflecting surface, a very distinct echo is heard. Now this echo cannot arise from the first course of pulses that are reflected to the ear, because the breadth of the cave is so small, that they would return too quickly to produce a distinct sensation

from that of the original sound: it therefore is produced by those pulses, which, after having been reflected several times from one side of the grotto to the other, and having run over a greater space than 127 feet, arrive at the ear in considerable numbers, and not more distant from each other, in point of time, than the ninth part of a second.

22. Some philosophers * of the Continent maintain, that sounds move only in right lines from the sounding body, in the same manner as light is propagated from the sun: and that we hear it only in different directions, as we see light, *viz.* by reflection. This theory (which contradicts prop. 42. B. 2. Newton), they endeavour to support by the observation, that echos are heard

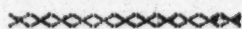
* Euler, Berlin Transf. an. 1763; Monf. Lambert, ib. Monf. Hube, Acta Erud. an. 1762, &c.

only in one direction; which, they say, could not be the case, if sound were propagated in all directions from the sonorous body. "The echo," says Monf. Lambert, "appears to me a decisive instance in this case. An echo is produced when sound is reflected from an obstacle at a sufficient distance. In cases of this nature, I have always observed, that the echo, so far from being heard all around, is heard there only, where the reflected sound passes according to the laws of reflection." (*See Berlin Transf. an.* 1763). But this argument is inconclusive, as we have shewn that the same effect would follow from sounds being propagated in all manner of directions, and that most probably the true reason why the echo of the sounding body, situated suppose at A (*fig.* 3), is heard only in one situation as at B, is not because the angles of reflection at C, D, E, F, &c. are equal to the angles of incidence,

(*see schol. prop. 17. B. 2. Ham. Con.*); but because the several sums of the lines $\overline{AC+CB}$, $\overline{AD+DB}$, $\overline{AE+EB}$, &c. are of the same magnitude (p. 60): whence it follows, that the pulses propagated from A, and reflected from the obstacles C, D, E, &c. will all meet in the focus B at the same instant, and consequently produce an echo; provided the interval of time which elapses between the perception of the original sound and the concurrence of the pulses at B, exceeds the ninth part of a second. So that an echo's being heard only in that point where the reflected sound passes according to the laws of reflection, is not a consequence of the equality of the angles of incidence and reflection, but of the equality of the spaces described by sound in equal times.

PART II.

ON MUSICAL STRINGS.



S E C T. I.

OF THE MOTION OF AN ELASTIC
FIBRE.

23. **I**F a non-elastic body AB receive an impulse in C, (*fig. 4*) a twofold effect will be produced; for the body will not only go forward in the direction of the stroke, but the surface AB will also be indented at E; the non-elastic fibre AB assuming the figure AEDB: which indenture will continue; the whole body in the mean time moving forward.

But if this body, and consequently the fibre AB, were elastic, the inflected part AED would restore itself to its former position, and produce an undulation or serpentine motion in AB; the whole body, at the same time, moving forward as before.

24. The manner in which this undulatory motion is propagated, is as follows: The impulse being made at C (*fig. 5*), at the same time that the whole fibre moves forward as before, the point C will relatively move faster forward, and drawing after it equal parts of the fibre on either side, will form the isosceles triangle AED. The point D will next, relatively, be set in motion, because it is acted on by two tending forces in the directions ED, DB; and therefore will ascend in a line that lies between these directions. But because AE is very nearly parallel to AB, the angle AED will be very

nearly equal to EDB, and the forces in the directions of the sides of those angles are equal, being the same tending force of the string. Therefore D will begin to ascend with the same force, very nearly, with which E begins to descend. Now as E begins to descend and D to ascend, at the same time, with equal forces, when E has descended to C, D will have ascended to F, (*fig. 6*), nearly as far from the axis as E, the fibre assuming the form ACFGB. But E, by the velocity acquired in its descent to C, will pass on to the other side of the axis, suppose to K, (*fig. 7*); F in the mean time will have descended, suppose to D, and G ascended to I; and the fibre will have assumed, nearly, the figure AKDIHB. The first wave therefore has now generated another contiguous to it, whose convexity lies on the opposite side of the axis; in the same manner, the wave DIH will generate another: and

thus will an undulation be carried on throughout the entire length of the fibre, the convexities of the contiguous waves always lying on opposite sides of the axis; while, as before, the whole body, collectively considered, is going forward in the direction of the original impulse.

Hence it appears, that if a tended elastic fibre, all the parts of which are equally susceptible of motion, receive an impulse, an undulatory or serpentine motion will be produced therein, at the same time that the entire fibre is carried forward in the direction of the impulse.

25. If now the extremities A, B be fixed, and the fibre receive an impulse at C, an indenture will be made at C as before, and, at the same time, every point of the whole fibre will have a tendency forward as before; but the

two points A, B alone being fixed, these alone will continue at rest, and therefore the other points of the fibre advancing forward, will cause the whole fibre to become concave to the axis AB. And after it has attained the utmost limit of its progress, its elasticity will cause the whole fibre to return to its former position; and by the velocity acquired in its return, it will pass on as far on the other side of the axis AB, or rectilineal position.

Hence will arise a slower vibration of the whole fibre by the first of the two effects of the impulse mentioned in (23), while at the same time, as before, the minute, rapid undulation, caused by the other of these effects, is carried on throughout its entire length; the string assuming a figure somewhat like that represented in the eighth diagram.

26. From what has been said it appears, that as every wave endeavours to generate another contiguous to it, whose convexity shall lie on the opposite side of the axis; the first wave endeavours to generate a series of equal waves. For as the time of the vibration of any wave is as its length, (*cor. 7. prop. 24. Smith's Harmonics*), if these waves were unequal, the times would be so too, and consequently the convexities would not successively lie on opposite sides of the axis.

27. If the impulse on any point of the string be so great as that, during the progress of that point, it shall agitate the part of the fibre which lies between that point and the nearer end of the fibre, it is evident that the latitude of that wave, and consequently of every subsequent one (26), will be double the distance of the point of impulse from the nearest extremity of the

string. If half the latitude of the first wave be an aliquot part of the string, an even and regular undulation will be propagated through it; and if it be half the whole string, there will not be any undulation at all.

If half the latitude of the first wave be an aliquant part of the string, after the motion has been propagated to the farthest extremity, there will be a new series of less waves, recurring in a contrary direction. Thus, if AD be not an aliquot part of AB (*fig. 9*), after the waves have been propagated to K, KB being then set in motion, will excite a new series of recurrent waves from B towards A: and if KB be not an aliquot part of AB, there will be a third series of waves less than either of the former propagated back again from A to B, and so on *ad infinitum*. Now as musical strings are so very much tended, we may always sup-

pose, that the impulse on any point is sufficiently strong to agitate the portion of the string between it and the nearest extremity of the string, during the progress of that point; and therefore we may conclude, that the latitude of the waves, in any series, is equal to double the distance of the point of impulse from the nearest extremity of the string.

28. In order to find whether this reasoning accorded with experiment, I procured as long a string as I conveniently could; and straining it tight, but not so much as to render it musical, I made the following remarks on its vibrations:

When I pulled it by a point near either of the extremities, and accurately examined its appearance during its vibration, I observed two forms of a chord, one vibrating very rapidly, while, at the same time, the other ap-

peared to roll slowly backwards and forwards, and to cross the former. Now those situations in which the chord appeared well defined, were certainly the extremes of its progress and regress, where having lost its entire motion, it must have continued longer than any where else. And as there were two appearances accurately defined, one moving slower, the other quicker, it evidently follows, that the chord must really have had two motions, one slower, the other quicker, according to (25). Sometimes, by striking the chord at random, I have seen three or four of these apparent chords crossing each other with various velocities (27). But when I pulled it by the middle point, there appeared but one image of the chord. When I pulled it by the middle of an aliquot part, I was not without hopes of being able to distinguish the waves of the undulation, from knowing where they would

intersect the axis, and accurately observing it in those points: but I never could discover any such intersections. However, the experiments, to me, appear as decisive in favour of the theory, as could be expected in a case where motions so rapid and subtle were the object of examination.

29. When the impulse is made only on one point of the string, and the motion thence propagated through the whole, the undulatory motion is such as we have described in (24): every wave generating another contiguous to it, whose convexity shall lie on the opposite side of the axis (*ib.*) But if several impulses be made on the middle points K, C, D, E, (*g.* 10), of the aliquot parts AG, GH, HI, IB, at the same instant, and in the same direction, the series of waves will be converted into a series of indentures, whose convexities will all look the same way;

and, at the same time, each impulse will contribute to produce a vibration of the whole string (25). For these impulses being equal, they will produce equal indentures in the string; and as the tension is equal throughout the string, these indentures will vibrate in the same time, and consequently their intersections G, H, I will always continue in the same points: at the same time that the whole string, collectively considered, has a motion forward as before; assuming nearly the figure represented in the 11th diagram.

S E C T. II.

OF SYMPATHETIC TONES.

30. **W**E now proceed to consider the phenomena of musical strings, and to shew how they are deducible from the principle which we have been hitherto endeavouring to establish. But I am far from maintaining all that is here advanced to be perfect and rigid demonstration; indeed the subtilty and intricacy of the subject seem scarcely to admit of it. However, the simplicity of the principle, as delivered in the first section of this part; the ease with which all the different phenomena, treated of in the following sections, may be derived from it; and the

variety of experiments which conspire to establish its certainty as far as could be expected in such an enquiry, will, I hope, leave little doubt of its truth.

Let the two strings AB, CD (*fig. 2*) be of equal diameter, length and tension; and let one of them AB be struck, so as to produce a forcible tone; the other CD will be so agitated by the pulses of the air, as to produce a tone; which consequently will be in unison with the former. For, the pulses of the air vibrate in the same time with the sounding body which produces them (12). Now these pulses propagated towards CD, will strike upon every point of it, at the same time and in the same direction: that which strikes on the middle point of CD will agitate the whole string; all the other impulses will produce a double effect, both a vibration of the whole string, and an indenture, according to (25). That

which strikes nearly on the middle of Cn will agitate that portion, forming nearly an isosceles triangle whose base is Cn ; at the same time another equal pulse will agitate the portion nD ; and these two indentures, or bellies, being always equal, their intersection will always continue at n ; and they will consequently vibrate in half the time of the whole string. In the same manner, three pulses will strike nearly, or accurately, on the middle points of three aliquot parts of CD , which will produce three intersecting indentures; and their nodes or intersections will also continue in the same points; and they will vibrate in one-third of the time of the whole vibration. And in like manner we must reason concerning all other aliquot parts.

Now of all these motions we are to consider which will predominate: and

that evidently must be the motion of the whole string. For all the subordinate vibrations of the aliquot parts must comparatively be destroyed, because they are performed in times different from that of the pulses; but the whole string vibrating exactly in the same time with them, every succeeding impulse from the air will conspire in direction with the motion already generated; and therefore will at length so far increase the motion of CD, as to render it sufficiently powerful to affect the ear. And besides, as the lengths of these indentures are less than that of the entire string, the vibrations produced in them will be proportionably less powerful (15). As to those pulses which strike on the middle points of aliquant parts of the string, they can never produce any sensible effect. For suppose CR (*fig. 12*) an aliquant part of CD; from the extremity C take CR as often as possible in CD, suppose five times,

there will remain the portion PD less than any one of these; the pulses that strike on the middle points of these equal portions produce equal effects; take from the other extremity D, the same number of equal parts, the pulses that strike on the middle points of these parts produce equal effects with the former; but as PD is less than QP, this series of indentures will intersect and break the former; and as the two series of indentures are equally powerful, they will destroy each other's effects. Besides, as these aliquant parts are incommensurate to AB, the times of their vibrations will be also incommensurate to that of the pulses; and therefore their vibrations will be extinguished by these pulses.

31. The more acute the tones of the strings AB, CD (*fig. 2*), the more difficult it is to excite a tone in the quiescent CD: for in order to produce

tones of equal strength in different strings, the impelling force, *cæteris paribus*, must be in the subduplicate ratio of the tending forces; that is, directly as the number of vibrations performed in a given time (15). So that to produce an octave requires a double effort; to produce a twelfth, a triple effort; a seventeenth major, a quintuple effort; and so on: conformable to what Newton tells us in the scholium of prop. 50. B. 2. Prin. "that the more rapid tremors are with greater difficulty excited in quiescent objects, by the pulses propagated from sounding bodies."

32. If the length of CD be but a little less or greater than that of AB, still the strong vibrations of AB will cause it to produce a tone; because if the tension and diameters of homogeneous chords be equal, the times of their single vibrations are in the direct ratio

of their lengths (*cor. 7. prop. 24. Smith's Harmonics*): therefore, in this case the times will be very nearly equal, and consequently the motion of the pulses will very nearly conspire with the motion of CD.

33. But if the string CD be considerably different in length from AB whether greater or less, and also incommensurable to it, the strong vibrations of AB will not excite any sound in it, because the times of the vibrations of the pulses and of CD will be so discordant, that every successive pulse will counteract and check the motion which the preceding pulse had generated. Neither, when CD is greater than AB, will its submultiple parts be sufficiently agitated to produce a tone, if these submultiples, which approach nearly to a common measure of the two strings, be very numerous: both because these parts are not commensurable to AB;

and because, by being so numerous, they will receive but very few conspiring impulses; as will presently be explained at large.

34. If the string CD (*fig. 13*) be exactly double of AB, and AB be so struck as to produce a forcible tone, the chord CD will resolve itself into two equal parts; each of which will separately vibrate, and produce a tone in unison with that of AB; the middle point E remaining very nearly at rest.

This experiment, together with some of those which follow, is mentioned by Wallis in his Algebra, p. 466. vol. 2. and was communicated to him by Narcissus Marsh, archbishop of Cashel, in the year 1676, as a new discovery made by Mr. Noble and Mr. Pigot of the University of Oxford. Long after, Monf. Sauveur communicated it to the

Royal Academy at Paris as his own discovery ; but upon his being informed, by some members present, that Doctor Wallis had published it before, he immediately resigned all pretensions to the honour of it *.

This phænomenon is to be accounted for in the following manner : the pulses propagated from AB, which strike on CD, produce a double effect, as in other cases, *viz.* a motion of the entire string, and also a series of indentures (29) ; the former of these two effects must be destroyed, for CD, being double of AB, will perform half a vibration, while AB performs an entire vibration. Therefore when CD has moved from E to F, and returned back to E, the pulse, which set E in motion,

* Sir John Hawkin's History of Music, vol. 3, p. 134.

will have performed an entire vibration; and therefore will be in its state of greatest condensation going forward, when CD is in the middle point of its regress; and consequently will totally destroy the motion of E , the middle point of CD , the impulses being equal and contrary: for the elastic force with which the string returns from F to E , will be equal to that with which it set out from E , that is, to the force of the pulse. And the same may be said of all the other impulses, as to the first of their two effects. Therefore the point E will very nearly continue at rest, the motion generated in it by each pulse being destroyed by that immediately succeeding. The pulses which strike against m and n , the middle points of CE and ED , produce the similar and equal indentures CdE , Efd ; which, because their latitude is double Cm (24), will intersect each other in E , the middle point of the string; and as the mo-

tions of these indentures, or submultiple strings, are exactly alike, they will constantly have the same intersection E. But farther, since these submultiple strings are of the same length with AB, they will vibrate exactly in the same time with it; and consequently every succeeding pulse will encrease the motion of these parts; and as no other aliquot parts of the whole string vibrate in the same time, there cannot be any other equally powerful series to cross and break them (30); and consequently their motions will, at length, become strong enough to affect the ear. The pulses which strike on the middle points of Cm, mE, En, nD, agitate those portions; but as they vibrate in half the time of the pulse, they will only receive an impulse every second vibration; and therefore the motions of CE, and ED will predominate. And besides, as the lengths of the aliquot parts Cm, mE, En, nD are less than those of

CE and ED, the successive pulses will generate in them a less degree of motion (15), and consequently these portions will not produce any sensible effect. The same may, *a fortiori*, be said of all other submultiples of CD. Though indeed we can conceive it possible, that the human ear might have been endued with such a degree of sensibility, as to be capable of hearing all these sounds at once *. The remaining pulses strike

* But had this been the case, we should never have had any idea either of harmony or melody; as every string contains in itself an infinite variety of discords. For since a string and its aliquot divisions are as the numbers 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c. the number of vibrations performed by them in a given time will be as the natural numbers 1, 2, 3, 4, &c. *Smith's Harmonics, prop. 24. cor. 7.* And consequently this series of sounds must contain all the possible varieties of intervals. If therefore every aliquot division produced a sensible effect by its vibration, we should hear in every musical string an infinite variety of chords, dissonant and consonant, in sharp and flat keys, at the same time. Thus would all the charms of melody be destroyed: and where many musical strings were found-

on the middle points of aliquant parts of the string ; and therefore every series of indentures produced by them will be crossed and broken by others equally powerful, and consequently will be extinguished, according to what we have shewn in (30).

ing together, this confusion of consonance and dissonance would be still farther increased, and therefore much more should we have been deprived of the perception of harmony. We have here therefore an instance of the admirable skill with which the different parts of nature have been adapted to each other by their all-perfect Contriver ; who as in other cases he appears to have consulted the welfare of his creatures, so from this instance we should infer, that he has not been less attentive to their innocent gratifications. Had the human ear been endued with a less degree of sensibility than it is at present possessed of, it is evident that we should have lost much of the delightful effects of harmony : and had it been endued with greater, we should have had no perception either of melody or harmony, as we have just now endeavoured to prove. It appears therefore, that the sensibility of the human ear has just attained the limit that contributes most to the pleasures of hearing.

Sir John Hawkins, in his History of Music, has complimented archbishop Marsh for his exquisite solution of this phenomenon; but I apprehend, that whoever shall be at the pains of examining into its merits, will be persuaded that he was induced to pass this favourable judgment on it, merely on the authority of Doctor Plott *.

35. It may perhaps be objected to the solution here given, that the point E is the vertex of the indenture dEf , and that as the angle dEf is equal to CdE , and the tending forces acting in the direction of the sides of these angles equal, the point E ought to vibrate through as great a space as that through which d or f vibrates. But in answer to this we are to observe, that the convexities of the indentures lie contrary

* See his Natural History of Oxfordshire, ch. 9.
§ 199.

ways, and therefore when d is going forward E is returning back; as has been explained at large in paragraph 24; and therefore when d receives a conspiring impulse from the air, E will receive the contrary; consequently the motion produced each preceding moment by the tending forces in the directions dE , and fE will be destroyed the next; whereas the motions of d and f are constantly increased. The true figure, therefore, of the string in vibrating, will be such as is represented in *fig. 14*; but as the motion of E is very inconsiderable, the points e and E may be looked upon as coincident, and the figure of the string may be represented by *fig. 15*. For the same reason that E is nearly quiescent in this case, all the nodes in other cases of indentures will be also nearly quiescent.

36. If, conversely, the string CD be forcibly struck, it will excite the tone

of AB; because, while CD vibrates once, AB vibrates twice, and therefore will receive a concurring impulse at the end of every second vibration.

37. Strike AB the octave to the fifth above the fundamental CD (*fig.* 16), and the base CD will resolve itself into three equal parts, and sound the unison of that octave. For AB is to CD as 1 is to 3, therefore the whole string cannot vibrate; for the particles of air adjacent to the string are supposed to impel it only when they are in their state of greatest condensation, that is, in the middle of their progress; therefore after the first impulse of these particles on CD, there will be another when they have performed an entire vibration, for then they will be in their state of greatest condensation again; and in the mean time CD will have performed one-third of its entire vibration, or $\frac{1}{3}$ YN; that is, if YN be tri-

fects in x and z , and YO in r and s , CD will have moved from Y to N and from N to z , and therefore will be returning backward whilst the pulse is going forward, and therefore its motion will be checked; when it receives the next impulse it will have got to s in its return, and therefore will be checked again; and at length the next impulse it receives will be when it has returned to Y. Since therefore during the entire vibration its motion is twice checked, and only once favoured by the pulse, it is obvious, that the whole string cannot be sufficiently agitated to produce a tone. It remains only therefore for us to consider the motion of the indentures produced by each pulse which strikes on CD.

The three equal pulses which strike on m , Y, and n , the middle points of the portions CE, EF, FD, will produce three equal intersecting indentures;

H

and as their motions are exactly alike, their nodes will constantly continue in E and F. But farther, these submultiple strings being of the same length with AB, they will vibrate exactly in the same time with it; and consequently every succeeding pulse will increase the motion of these parts; and as no other aliquot parts of the whole string vibrate in the same time, there cannot be any other equally powerful series to cross and break them (30); and consequently their motion will at length become strong enough to affect the ear. As to the motions produced in any other aliquot or aliquant parts of the string, they cannot produce any sensible effect, for the reasons assigned in (34).

38. If conversely the string CD be forcibly struck, AB will return its proper tone; because it will receive a concurring impulse at the end of every third vibration.

39. Strike forcibly AB the fifth (*fig.* 17) above the fundamental CD, and the tone generated, by the pulses, in the quiescent string CD will be the acute octave to AB. For AB is to CD as 2 is to 3; therefore while AB performs one vibration, CD will perform two-thirds of a vibration, or $\frac{2}{3} \times 4$ YN or $\frac{8}{3}$ YN; that is, if YN be trisected in x and z , and YO in r and s ; CD will describe eight of these little parts, while AB vibrates once; therefore when CD receives its second impulse it will be in s , returning backward whilst the pulse is going forward; and therefore will be checked in its motion; it will receive the third impulse when it is in z , returning also backward whilst the pulse is going forward, and therefore will be interrupted again; lastly, it will receive a conspiring impulse at the end of the third vibration of AB, and the second of CD. Therefore, as in

(34), the whole string cannot be sufficiently agitated to produce a tone.

It remains therefore only for us to consider the motion of the several series of indentures in the string: the pulse propagated from AB, which strikes on m the middle point of Cp , would excite a vibration in the string Cp equal in length to AB, which would in every vibration of AB receive a new impulse, and consequently would predominate, were it not that another equally forcible pulse striking on p the middle of mD , will produce an equally powerful indenture at the other extremity, which will interrupt and break the former, because they are aliquant parts of the whole string (30). The portions therefore of CD which will produce the predominant tone, must be aliquot parts of the string (*ib.*) But they must also be such as will receive the greatest number of conspiring impulses from

the aerial pulses propagated from AB. The length therefore of these aliquot parts of CD must be a submultiple of AB, that they may receive any concurring impulse; and that they may receive the greatest possible number, their length must be the greatest common measure of the two strings; that is, must be equal to half of AB, which produces its acute octave.

The coincidences in this experiment are only at the end of every second vibration of AB; in the former, they were at the end of every single vibration of the same string: hence we see why, in this case, the tone is more languid than in the former.

It might perhaps be objected, that since the parts Cm, mp, pD vibrate in half the time of AB, they should destroy its motion; for the same reason that the vibrations of AB, in paragraph 34

(*fig. 13*), destroy the motion of **CD**, its double: but that this is contradicted by experience. The answer to this objection is obvious; in paragraph 34, the vibration of **AB** is the cause of the motion of **CD**, and the impulses from the air on **CD** being equal and contrary, must destroy each others effects. But in the present case, it is the vibration of **AB** which generates the motion in the parts *Cm*, *mp*, *pD*; and as this generated motion is very much inferior in strength to that which generates it, it is evident that the pulses propagated from these parts can have but a very inconsiderable effect on **AB**.

40. If, conversely, **CD** (*fig. 17*) be forcibly founded, **AB** will be resolved into two equal parts, and will produce the same tone that **CD** produced before; that is, the acute octave of **AB**, or the twelfth above **CD**. For while **CD** vibrates twice, **AB** vibrates three

times ; and therefore while CD vibrates once, AB will perform one entire vibration and also an half ; therefore AB will receive the second impulse, from the aerial pulses, in the middle of its regrefs, and therefore its motion will be destroyed ; and the whole string will not produce a tone. But the pulses which strike on the middle points of An and nB agitate those parts, which therefore, being to CD as 1 is to 3, will receive a conspiring impulse at the end of every third vibration, without any interruption from the pulses in the intermediate time ; and also as they are aliquot parts of the whole string, there will be no series of indentures to cross and break them (30) ; consequently their motion will predominate, and become powerful enough to affect the ear.

41. From these experiments we may derive this general conclusion, that if

two commenſurable ſtrings be of equal diameter, density, and tenſion, and one of them be forcibly agitated; the tone produced in the other will be that of ſuch a portion thereof as is the greateſt common meaſure of the two ſtrings. And as the vibrations of all uniſons are performed in the ſame time; the tone produced in any ſympathetic ſtring by the pulſes propagated from another, will be the ſame as in the former caſe, although the ſtrings be of different diameters, densities or tenſions, provided they be tuned in uniſon with ſuch as agree in thoſe circumſtances.

42. Malcolm has been betrayed into a miſtake with reſpect to theſe tones, in ſuppoſing that an acuter ſtring cannot excite a tone in a graver: “ In ſome caſes, ſays he, the motion of the untouched ſtring is ſo checked as never to be ſenſible, or at leaſt to give any ſound;

and in fact we know, that in no case is this phænomenon to be found but the unison, octave, and fifth; most sensibly in the first, and gradually less in the other two, which are also limited to this condition, that the *graver* will make the *acuter* sound, but not contrarily." (*See his Treatise on Music, p. 87.*) Now on the contrary, we find by experiment in (39) and (40), that if the lengths of two strings be to each other in the ratio of 2 to 3, in which case the one is a fifth to the other, the same tone will be excited in the quiescent string, whether it be the *acuter* or the *graver*. But for the greater security of success, the strings should be tended on the same instrument.

Rousseau has spoken of these sympathetic tones in his Musical Dictionary, (*see article Unisson*); and has shewn how motion may be communicated from one string to another which is a submulti-

ple of it : but he has not shewn, conversely, how motion may be communicated from the acuter to the graver ; much less has he explained how the sympathy is produced where the strings are to each other, not as an integer to unity, but in any commensurable ratio whatsoever.

43. The theory here delivered will enable us to decide on the origin of the minor mode, as given by Rameau. Let there be three strings whose lengths are as the numbers 5, 3, 1, their density, diameter and tension being the same ; if the shortest of these be sounded, it will cause the other two to resolve themselves, the one into five, the other into three equal parts, each of which will tremble ; and we shall observe four quiescent nodes in the one string and two in the other. This trembling, M. Rameau says, discovers a certain natural connection between

the tones produced by the whole strings. He observes farther, that it makes no alteration in the substance of harmony, whether we use a certain note or any of its octaves. Take then the acute octave of the lowest of these tones, and the grave octave of the highest, and the three numbers expressing the chord will be 3, $\frac{5}{2}$, 2, which is the minor mode *. But the fallaciousness of this method of deriving it from the mere communication of motion in sympathetic strings, will evidently appear from what has been already demonstrated: for the same manner of arguing may be applied to various other cases, in which the resulting chord is dissonant. It has been shewn, that a string, forcibly sounded, will cause any other, of equal diameter, density, and tension, which is a multiple thereof, to resolve itself

* See *Elemens de Musique*, p. 23. and Antonio Eximeno *Dell' origine e regole della Musica*, p. 97.

into such aliquot divisions as are equal in length to the submultiple string. Let then the lengths of three strings be as the numbers 7, 3, 1, the shorter of these being forcibly founded, will cause the other two to tremble and resolve themselves, the one into seven, the other into three equal parts. This trembling therefore, according to Rameau, intimates some natural connection between the tones produced by the whole strings. If now we take the acute double octave of the lowest, the three numbers expressing the chord will be 3, $\frac{7}{4}$, 1, which is a dissonance. In the same manner, let the lengths of the three strings be as the numbers 11, 3, 1; if now we take the acute double octave of the lowest, and the grave octave of the highest, the resulting chord will be denoted by the numbers 3, $\frac{11}{4}$, 2; which is also a dissonance. And the like may be proved in a variety of other cases.

S E C T. III.

OF SECONDARY TONES.

44. **I**T was observed by M. Rameau about the year 1750, and afterwards by M. D'Alambert and many other musicians, that a musical string, besides its peculiar and fundamental tone, produces several others which are called *secondary*. And on this single fact it was, that M. Rameau founded his ingenious system; whence he has been stiled *the Newton of harmony*; as he has derived the whole from one principle, to wit, the fundamental base; in the same manner as Newton, from the single principle of gravitation, has ex-

plained the most remarkable phenomena in physics *.

The fact is easily ascertained, by striking the base notes of an harpsichord or violoncello; for when the principal note has decayed and languished, you will hear distinctly the octave above the principal, the twelfth, and seventeenth major. This however is not a modern discovery, for it was known to the writers on harmonics at least so long ago as the year 1618. Mersennus was acquainted with these secondary tones, as may be seen in his chapter on the difficulties of music, *Opera Mathem.* vol. 1. p. 355: and Des Cartes, his cotemporary, in a letter to Mersennus, speaks of them in the following manner: "As to the different tones which are produced at the

* Sir John Hawkins's *History of Music*, vol. 5. p. 385.

same instant by the same chord, I know no other reason that can be assigned than that which, I think, I formerly communicated to you by letter, *viz.* that while the whole string performs its vibrations, its parts may have less sensible motions; which, meeting with the whole body of air already set in motion by the principal vibration of the entire string, must necessarily render the percussions which it makes on the ear double, or triple, or quadruple, or quintuple, and so generate the octave, twelfth, fifteenth, and seventeenth." *Ep. P. 2. Ep. 106.*

M. Bernouilli accounted for these secondary tones in a manner somewhat similar to that of Des Cartes, by supposing that every string resolves itself into aliquot parts, and that while the whole string performs its vibrations, the sub-multiples also perform their secondary vibrations. Thus the principal vibra-

tion produces the base note, that of one-half the string generates the octave, that of one-third the twelfth, and that of one-fifth the major seventeenth. This very ingenious hypothesis, was adopted by Euler, and after him by Antonio Eximeno, in his excellent treatise *Dell' origine e regole della Musica*, printed at Rome in the year 1774. Bernouilli and Euler, on the supposition that strings do resolve themselves in this manner into aliquot divisions, have given us various calculations of the forms which they assume in their vibrations. He that wishes to enter into this mathematical discussion, may find it at large in the Philosophical Transactions of Berlin and Petersburg. The grand result of their enquiries is, that the vibrations of a string, whatever form it assumes, are synchronal; and that the time of each vibration is proportional to the length of the chord, its diameter and tension being given. For if the

length of any uniform chord be called a , κ the weight of a portion of the string whose length is k , the tending force π , and the space described in a second by a body falling freely from a state of rest b , then the time of the vi-

bration will be $\frac{a \kappa^{\frac{1}{2}}}{\sqrt{2 b k \pi}}$, (*Acta Petrop.*

v. 17. p. 394). Consequently in the same chord, since all these are invariable quantities, the time of the vibration must also be invariable: and in different chords of the same diameter, density and tension, since $\kappa^{\frac{1}{2}}$ and $\sqrt{2 b k \pi}$ are given quantities, the times of their vibrations will be as a ; that is, directly as their lengths.

45. But these ingenious writers have not favoured us with their conjectures why musical strings should thus resolve themselves into aliquot parts; nay even their own theories do not seem very well to consist with such an idea. For

Bernouilli asserts, (*vide Phil. Transf. of Berlin. an. 1753*), "that the curve of a string in vibrating will be a trochoid, or companion of a cycloid, extremely prolonged;" in which he follows Mr. Taylor, who has delivered this doctrine in his method of increments *. But there is nothing in such a vibration which can possibly lead us to conclude, that the string will resolve itself into submultiple trochoids. And M. D'Alembert has very properly made this objection to Bernouilli: "I demand of him first, says he, what is the reason that a vibrating chord should be composed of several trochoids? And secondly, which he has explained still less, what is the cause that determines these trochoids to be constantly such as to produce the octave, twelfth, and seventeenth, rather than any other sound?" (*See Encyclopædie, article Fondamental.*) Euler, who is more general in his solutions,

* See also Smith's Harmonics, sect. 11.

has given us a scale for calculating the various figures which a string will assume in vibrating, the figure of the string before vibration being given; and particularly, if a string be drawn into a triangular form ACB (*fig. 18*) before vibration, he proves, that the point C will so descend towards the base AB, as that the string will always assume the figure AFGB, FG being parallel to AB. (*See Acta Petrop. an. 1772*). Hence we should conclude, in a case of this kind, since it does not appear that the string would be resolved into aliquot parts, that we could never hear a secondary tone; which is contradicted by experiment. And the same objection will hold, whatever be the form of the string before vibration.

46. The author of the *Elucidationes Physicæ* upon Des Carte's Compendium of Music, and after him Monf. Mairan,

(*See Encycl. Yverdon, article Son*), has attempted to account for these tones on different principles. He supposes that there are in the air particles of every possible variety of tension, which by their vibrations produce all the indefinite variety of notes ; that some of these will of consequence vibrate quicker, others slower : and that when a string is vibrating, it communicates its motion principally to those particles which are of the same degree of tension with itself, and consequently vibrate in the same time. After these, the particles which least interrupt the chord, and which are least interrupted by it, are such as perform two vibrations in the same time that the chord performs one ; because their vibrations begin together after every second vibration ; these produce the acute octave. In the same manner, the twelfth and seventeenth are generated by the particles which vibrate three and five times while the chord vi-

brates once ; each tone decaying in strength according to its acuteness, because less favoured by the vibrations of the chord.

But this theory is liable to every species of objection ; for it not only appears to be a most arbitrary hypothesis, devised merely with a view to say something on this intricate subject, without any regard to the real constitution of nature ; but is in fact contradicted by experience. For if these tones depended on the configuration or tension of the particles of air, our voices, in singing, should as frequently produce them as stringed instruments. But besides, without recurring to experience, we can from theory also demonstrate that this hypothesis is false ; for it would follow as a direct consequence, that admitting the air to be a fluid perfectly homogeneous, all musical strings, however different from



each other in length, diameter, density and tension, would produce the very same tone: now on the contrary, Sir Isaac Newton has demonstrated, (*Prop.* 47. *L.* 2), that the pulses vibrate according to the law of the sounding body; and therefore that the tone will vary according to the number of vibrations performed by that body in a given time, supposing the medium homogeneous.

47. *Monf. D'Alembert* has advanced another objection to this hypothesis; and although the opinion which he controverts be false, as I think we have sufficiently proved, yet it is necessary I should endeavour to obviate this objection, as it would equally hold against the doctrine which I shall presume to offer upon this difficult question.

He says, that even if we should admit such particles to exist, the proposed

effect would not follow from the coincidences of the pulses of the air with their vibrations : for as an octave string vibrates twice for one vibration of the base, it will be as often opposed as favoured by the vibration of the base. If a string be one-fifth of the principal, it will be opposed twice and favoured three times ; thus suppose CD (*fig.* 19) one-fifth of AB, while AB vibrates once, CD will vibrate five times, therefore while the pulse is going forward through a space equal to EF, CD will move twice from G to H and from H to G ; and besides, will also move once from G to H ; but during this time, according to D'Alembert's reasoning, its motion while going forward from G to H will be aided by the pulse, and opposed while returning from H to G ; that is, it will be aided three times in running over the space GH, and opposed twice. The same thing will happen during the other half of the vibra-

tion of AB ; therefore during the whole vibration, it will be favoured by the pulse while it describes the space GH six times, that is, during three vibrations ; and will be opposed four times, that is, during two vibrations. And in general any submultiple string whose denominator is an uneven number, will, while the multiple string vibrates once, be opposed during the less, and favoured during the greater of the two portions of time into which the whole time is divided, which portions are denoted by the two most nearly equal integers into which the whole number of vibrations can be divided. Hence, he says, the octave ought to be heard less distinctly than the twelfth or seventeenth, or rather not heard at all: contrary to experience. *See Encyclopædie.*

But Monf. D'Alembert seems to me to have here fallen into an error, in his manner of conceiving the effect of the

pulses : for we are to consider the particles of air as giving their impulse only when they constitute a pulse, that is, when they are in their state of greatest condensation, or in the middle of their progress; according to our reasoning in (12) *. Now in the case of the octave, the first impulse made on AB (*fig.* 13), by the pulses propagated from CD, will be when the adjacent particle is in the midst of its progress; and it will not receive another till that particle has got into the same position again: and as AB performs exactly two vibrations while CD performs one, it is obvious, that when the particle gets to the middle of its progress, AB will be in the instant before the middle of its progress; and consequently will receive a conspiring impulse as before. Therefore AB, instead of being as often opposed as favoured by the pulse, will not be opposed at all. And the same mode

* See also Smith's Harmonics, sect. 1. art. 12.

of reasoning may be applied in the case of any other submultiple string.

48. The authors of the *Encyclopædia Britannica* have advanced a new solution of these phenomena: "With regard to the production of the different tones from the bass string of an harpsichord, say they, we can only offer a conjecture, which is, that as the strings of musical instruments are fastened at both ends and very tense, the vibrations of the middle parts must be performed much more easily than those towards the extremities; consequently, as vibrations must have a certain degree of strength before a sound is produced, the middle parts of the string may vibrate so as to produce a sound, while the extremities have lost that power. This will be equivalent to shortening the string, and consequently the tone must grow gradually more acute." (See

article Acousticks). Now were this hypothesis true, according as the motion of the string languishes, and its extreme portions become gradually quiescent, we should hear a continued succession of all the possible tones which are more acute than the original tone of the entire string; directly contrary to experience.

49. Since therefore the theories, hitherto advanced respecting these tones, appear to be unsatisfactory and imperfect, we must have recourse to some other principle for their solution. And perhaps none will appear more probable than that which has been already applied to sympathetic tones: for the simplicity observable in the works of nature, gives considerable force to an hypothesis which can be applied with equal facility to the solution of many different phenomena.

When a musical string performs its vibrations, it excites pulses in the air which are propagated in all directions; these striking against all the surrounding bodies, every point of impulse becomes a new centre, from whence pulses are also propagated in every direction; some of which will consequently be propagated towards the sounding body, and will strike along its entire length, in the same manner as if they had been propagated from another string in unison with the proposed one. Thus will the present case become exactly of the same nature with that mentioned in (30). The pulses which strike against the middle point of the whole string, and which at the time of their impulse on the string are moving in a contrary direction to it, will only diminish the quantity of motion of the entire string: those which strike against the middle points of the halves, will endeavour to agitate these

halves, the motion being communicated equally on each side of the point of impulse (24); and therefore will not be interrupted, because they are aliquot parts of the string (30). Now as the time of the vibration of the halves is half the time of that of the entire string, the pulse generated by the whole string will coincide with every second vibration of the halves, and therefore will not interrupt them, and consequently will, at length, increase their motion so much as to enable them to produce a tone. Those particles which strike against such points of the string as are not the middle of aliquot parts, will excite vibrations that will both be interrupted by other series equally powerful, and by the original pulse of the whole string, and therefore will produce no sound (*B.*) Those pulses that strike against the middle points of each of the third parts of the string will agitate these parts, which will receive a concurring

impulse at the end of every third vibration, and therefore their vibrations may at length become strong enough to affect the ear. And the like holds of other aliquot parts ; but the shorter the portion, the fewer will be the coincidences of the pulses with the vibration of that part, and consequently the weaker the tone. The greater the tension of the fundamental the more difficult it is to hear the secondary tones, and the difficulty increases in the direct ratio of the number of vibrations performed in a given time by the fundamental. For to produce tones of equal strength in strings of the same length, diameter, and density, the impulse must be in the subduplicate ratio of the tending force (15) ; that is, reciprocally as the time of the vibration, or directly as the number of vibrations performed in a given time. And if the strings be of equal diameter, density, and tension, but their lengths different, a given

number of equal impulses will produce a stronger tone in that which is the longer. (*Ib.*)

The phænomena therefore of secondary tones appear to be precisely the same with those of sympathetic tones; and are to be accounted for on the very same principle: they are always the octave, twelfth, fifteenth, and seventeenth major; because these notes are produced by the half, third, fourth, and fifth parts of the entire string; they gradually decay in strength, according as the tone is sharper, because the number of concurring impulses diminishes, as the number of the aliquot divisions increases; and we hear them most distinctly in the lower strings of the violoncello or harpsichord, because the length is greater or the tension less in these strings than in the higher.

50. Besides those tones mentioned above, M. de Betizhy, on the testimony of Rameau, affirms that three others are also perceived ; namely, the octave of the twelfth, a sound which he called *the lost sound*, and the triple octave of the fundamental, (*see Exposition de la Theorie & de la Pratique de la Musique, Part. 2. Cap. 2. Art. 1.*) Now the first of these was produced by one-sixth, the second by one-seventh, and the third by one-eighth part of the entire string.

51. As the theory of sympathetic tones (43) enabled us to discover Rameau's error in deriving the minor mode from the mere communication of motion to quiescent strings ; so the phænomena of secondary tones will shew us, that D'Alembert has fallen into a similar mistake, in his account of the origin of both modes, major and minor ; which he has endeavoured to

derive from these secondary tones in the following manner :

The fundamental tone *ut* of any sonorous body being sounded, we find it always accompanied by its twelfth, and seventeenth major, that is, by the octave of *sol*, and the double octave of *mi* : the most perfect chord therefore which can be formed with any note is that which consists of these secondary tones; denoted by the numbers $\frac{1}{2}$, $\frac{2}{3}$, since this chord is the work and choice of nature. Now as it makes no alteration in the substance of harmony whether we use a certain note or any of its octaves, we may substitute the grave octave of the twelfth, and grave double octave of the seventeenth instead of these tones themselves : the resulting chord therefore *ut*, *mi*, *sol*, will be denoted by the numbers 1, $\frac{4}{3}$, $\frac{2}{1}$, which is the major mode.

This argument is founded on the hypothesis, that those tones which are naturally connected, form the most pleasing consonance. But the fallacy of this principle is evident from the testimony of M. Betizhy, who informs us, as we have already observed, that a musical string produces, together with the twelfth and seventeenth, a tone which he calls *the lost sound*, because it destroys the harmony. It appears therefore, that the natural connection of the secondary tones, is no proof that they will form the most pleasing consonance ; but that it is a mere mechanical effect, to be accounted for on physical principles, as we have attempted to do in the former part of this section.

The minor mode he derives from the major, in the following manner : in the modulation *ut, mi, sol*, which is the major mode, denoted by the numbers $1, \frac{4}{3}, \frac{2}{3}$, or the equivalent numbers as to harmony, $1, \frac{1}{3}, \frac{1}{3}$, the sounds *mi* and *sol*

are so proportioned one to the other, that the principal sound *ut* causes both of them to resound; but the second sound *mi* does not cause *sol* to resound, which only forms the interval of a minor third. Let us then imagine, that instead of the sound *mi*, another sound is substituted between the sounds *ut*, *sol*, which, as well as the sound *ut*, has the power of causing *sol* to resound, and which is, however, different from *ut*. Now a string which is quintuple of that which produces *sol*, will have this effect; if then we substitute the tone of this string instead of *mi*, the resulting chord will be denoted by the numbers 1, $\frac{5}{3}$, $\frac{1}{3}$, or by the equivalent numbers as to harmony, 1, $\frac{5}{6}$, $\frac{2}{3}$; which is the minor mode.

The harmony therefore of the minor mode, according to D'Alembert, arises from this, that both the first and second tones of the chord cause the other

to resound. But that this is an insufficient account of the matter will appear from observing, that the same effect may take place in other chords which are dissonant. We have seen before, that any string, forcibly sounded, generates besides the twelfth and seventeenth major, a sound which is called *the lost sound*, produced by the seventh part of the entire string. If therefore we take a string sevenfold of that which produces *sol*, it will cause *sol* to resound; substitute therefore the tone of this string instead of *mi*, and the three numbers denoting the chord will be 1, $\frac{2}{3}$, $\frac{1}{2}$, in which both the tones denoted by 1 and $\frac{2}{3}$ cause the other to resound; substitute now for the intermediate tone its double octave, and the resulting chord will be denoted by the numbers 1, 1^2 , $\frac{1}{2}$, which is a discord, although possessed of those qualities which render the minor mode harmonious. In the same manner, a string elevenfold of $\frac{1}{3}$ would

cause it to resound, if our organs of hearing were a little more sensible than they are, consequently the chord $1, \frac{1}{2}, \frac{1}{3}$; or, which is the same thing, the chord $1, \frac{1}{2}, \frac{2}{3}$ is possessed of those qualities which render the minor mode harmonious, and yet is discordant. The origin therefore of the modes, according to D'Alembert, however ingenious, is certainly fallacious, as we have now proved; and is farther made evident by Antonio Eximeno, in his admirable treatise on Music, where he has shewn, that a close adherence to the principles of the French musicians is attended with infinite violations of harmony; and that these ingenious writers could not have given us such excellent rules for practice, if these rules had not destroyed their theory *.

* Ant. Eximeno, Lib. 1. Cap. 5. § 5.

§2. D'Alembert has objected, that the secondary tones cannot arise from any submultiple divisions of the string, because the extremities of these parts are moveable; which is not the case of musical strings: "We might easily conceive, says he, how a chord should produce, besides the principal sound, the octave, twelfth, and seventeenth, if the points of the chord which form the extremities of the aliquot parts were really nodes or fixed points, so as that the parts of the chord comprised between the nodes, should perform in the same time, the first, one vibration; the second, two; the third, three; the fourth, four; the fifth, five, &c. In this case we might consider the string as composed of five different parts, placed in a right line, each immoveable at both its extremities, and forming by their different lengths this series or progression, $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5},$ &c. But experience demonstrates that this is not the case. In

a chord which performs its vibrations freely, we observe no other nodes, or points absolutely at rest, except the extremities." *Encyclopedie, article Fondamental.*

But it is not necessary that a musical string should have its extremities fixed and immoveable, as will appear from the following experiments: To a hook A (*fig. 20*) was fastened a wire string AB, and from the extremity B was suspended a weight sufficiently heavy to render it musical; I then fastened the point B, but in such a manner as that the weight should be totally exerted in tending AB; I then struck the string and observed the tone: then letting the weight hang freely, I struck it again, and though the weight vibrated without any interruption, yet still the note continued the same as before. To prove the same thing when both ends were in motion, I screwed on a plate of metal

at C, which stopping the vibration there, the musical string was reduced to the length CB; which produced the same note whether both the extremities C and B were fastened, or permitted to vibrate freely, by the impulse made on CB. Hence we see, that whether one, or both ends of a musical string be moveable, it will still produce the same tone as when they are both absolutely at rest, provided its length and tension continue the same.

53. But M. D'Alembert proceeds to object: "It is true, says he, that considering the nodes as moveable, and also the vibrating chord as composed of several trochoids, the different points of this chord perform their vibrations in different times. But it is easy to see, that this difference of vibrations cannot serve to explain the multiplicity of sounds. In fact, let us suppose, for the greater simplicity, and the easier

conception of this matter, that the chord forms only two equal trochoids, so as that the middle point of the chord shall be the common extremity of the two trochoids; we admit that whilst this middle point of the chord performs one vibration, the middle point of each of the submultiple trochoids will perform two: but it is easy to see, that each of these vibrations will not be performed in an equal time, and therefore that the reunion of the two vibrations cannot produce the octave of the principal sound, which is given by the middle point of the chord. We conclude therefore, that the different vibrations of the different points of the string are insufficient for the explication of the variety of sounds which it produces. This is not all: if the middle point of the chord performs one vibration, whilst the middle points of each of the trochoids perform two, it is easy to see, that the other points will

partake more or less of the law of the motion of these two, according as they are nearer to, or farther from them. Thus, properly speaking, the law of the vibrations of each point will be different, and each should produce a particular tone, which, by its mixture with the others, must generate a confused harmony, and kind of cacophony. Why then, says he, does not this happen? And why does the ear distinguish those tones only, in the sound of the string, which form a perfect concord?" The former part of this objection has been obviated by Bernouilli, in the Berlin Transactions for the year 1755, and he has there shewn, that D'Alembert has fallen into an error, in not referring the time of the vibration of the subordinate strings to their moveable axis. Euler has also demonstrated, as we have already observed (44), that the vibrations of a string, whatever form it assumes, are isochronal; conse-

quently, that whatever be the form of the subordinate trochoids, each of their vibrations must be performed in an equal time. But without having recourse to this abstract reasoning, we can evidently see, from the last mentioned experiment, the inconclusiveness of this objection: for it is there proved, that the tone of a string is not affected by the motion of its extremities; and therefore if the whole string be resolved into two equal parts, they must generate the acute octave of the entire string, whether it performs its own slower vibrations at the same time or not.

The latter part of the objection, which denies the possibility of a point's performing different species of independent vibrations at the same time, is overturned by our observation, that the human ear is capable of accurately distinguishing many different



sounds which strike it at the same instant; whence it follows, that the fibres, by which these sensations are conveyed, must be capable of many different vibrations at the same time; otherwise a multiplicity of concurring sounds would excite but one confused sensation, resulting from their compounded effect. And that several different species of vibrations may actually exist in the same body at the same time, undisturbed and absolutely independent of each other, may thus be demonstrated. But observe that I speak here only of the acceleration of points, according to Sect. 2. L. 2. Prin. not of the mutual attraction of bodies.

Let the point D (*fig. 28*) be attracted towards the point T with a force varying in the direct ratio of the distances DT, and D will describe an ellipsis whose centre shall be in T, (*cor. 1. prop. 10. lib. 1. Prin.*) Let now a third

point S attract the two points D and T, with accelerating forces, varying according to the same law. The force SD, (by *cor. 2. Laws of Motion*), is resolved into the forces DT, ST. The force DT by which D is urged towards T, being proportional to the distance, the point D will still continue to describe an ellipsis whose centre shall be in T, as before, but with a quicker motion. (*See prop. 64. lib. 1. Prin.*) The remaining accelerating force ST acting on D, and the accelerating force ST which acts on the point T, attracting those points equally, and in directions parallel to ST, do not at all change their situations with respect to each other, but cause them to approach equally to a line which we may imagine drawn through the point S, and perpendicular to the line TS. But that approach will be hindered by giving the whole plane in which D performs its motion a projectile impulse, in

which case the motion of D with respect to T will not be disturbed, (*cor. 5. Laws of Motion*), and the point T will describe an ellipsis whose centre shall be in S, (*cor. 1. prop. 10. Prin.*) Let now a fourth point V attract the points S, T, D with forces in like manner, directly proportional to the distance, and by a like argument it will be demonstrated, that the point S will describe an ellipsis whose centre shall be in V; the motions of D and T remaining the same as before. And by the same method more ellipses may be added at pleasure: thus will the point D perform several different periodical revolutions at the same time, the first with respect to T as its centre, the second with respect to S, the third with respect to V, and so on. Let now the minor axes of all these ellipses vanish, the major axes remaining unvaried, and the different periodical revolutions will be converted into as ma-

ny different vibrations, which will be performed according to the law of a cycloidal pendulum, (*prop.* 38. *lib.* 1. *Prin.*) that is, according to the law of a sounding body (12). It appears therefore, that a point of a musical string may perform several different and independent vibrations at the same time; which was the thing to be demonstrated.

M. Bernouilli has given us the following simple demonstration of the same truth in the following manner: Let the body A (*fig.* 29) be attracted towards the point B with a force directly as the distance, this body will perform its isochronal vibrations with respect to that point, (*prop.* 38. *lib.* 1. *Prin. Math.*) Let us now suppose that, whilst A vibrates in this manner, the entire line in which it performs these vibrations is equally accelerated towards the point C, so as that the system

AB may suffer a common gravitation towards C, which may also be directly proportional to the distance BC; this gravitation will not disturb the motion of A with respect to B, (*cor. 5. Laws of Motion*), and the system will perform its vibrations in respect to C, according to the same law with which A vibrates in respect to B. We next may suppose that the three points A, B, C suffer in each instant an equal gravitation towards a fourth point D, proportional to the distance CD. In this manner we may multiply indefinitely the gravitations of the body A towards different centres, and it will thus perform an indefinite number of independent vibrations in respect to these different centres. *See Berlin Transactions, An. 1755.*

54. In these secondary tones we sometimes hear the twelfth and seventeenth, when we cannot hear the oc-

tave, though from our reasoning on this subject we might naturally conclude, that the tone of half the string should be heard more distinctly than any other secondary tone, because half the string receives a greater number of conspiring impulses from the air in a given time than any other of its aliquot parts. But in these feeble notes we cannot always discern the octave, because it produces so smooth a coincidence with the base note as not to be distinguishable from it. Thus we find, that in an harpsichord or organ accurately tuned, the various parts of the diapason or octave stop are not distinctly perceived.

55. We have hitherto explained these phenomena of musical strings from the principle of the aerial pulses, and shewn that the reason of their resolving themselves into aliquot parts is, that these parts vibrate in such times, as to

receive the greatest possible number of successive conspiring impulses from the air, and at the same time not to interfere with, and disturb, each others motions. Now to shew that this is the true cause, and not any property in a string of resolving itself into *equal* parts, I procured a wire string AB, (*fig. 21*), part of which, CB, I got drawn down to a considerably less diameter than that of the part AC: this compound string was fastened at A, and tended by a screw at B. By a moveable bridge I so divided the string at D, as that the unequal portions AD, DB were in unison. I then removed the bridge, and sounded strongly, on a violin, the note which each of these parts, separately, would produce; and the compound string returned the unison. I then placed a light slip of paper at D, another on the middle point of the portion AD, and a third on the middle of DB; and on sounding the

same note as before, the two latter papers fell off, and that at D remained on the string; which shewed that the string had resolved itself into two unequal, but unisonal portions, AD, DB. But independently of this latter experiment, I might have been certain, that the point D was comparatively at rest, for no other division of the string could have produced the note which was founded on the violin. Hence then we find a string resolve itself into two unequal portions; consequently there is no property in strings of resolving themselves into aliquot parts only, any otherwise than as those parts produce the same tone, and consequently vibrate in the same time, so as to receive the greatest possible number of conspiring impulses from the pulses of the air.

S E C T. IV.

OF ACUTE HARMONIC
TONES.

56. **T**HE phænomena of acute harmonic tones were, as far as I can learn, first exhibited to the public by M. Sauveur, who performed many experiments relative to them, before the Royal Academy of Sciences in the year 1701.

57. If you press your finger lightly at D, (*fig. 22*) on the string of a violin or violoncello, so as to divide it into two parts, which may be to each other as 1 to 2, and draw the bow gently over the string between D and B, you will

hear what is called an *harmonic* tone, which will be in unison with the note produced by one-third of the whole string; that is, it will be the octave to the fifth above the fundamental. M. Sauveur made this experiment.

The cause of the effect is this: the impulse made on *o*, the middle of DB, endeavours to produce not only a vibration of the whole string, but also an *undulation*, because the impulse is made only on *one point*, (25). The finger pressing on D prevents the vibration of the entire string; there remains therefore the undulatory motion only, which is propagated under the finger at D to the adjacent portion of the string. And as in this case the effect is merely an undulation, the motion being communicated from one extremity of the string to the other, the first wave will endeavour to generate a series of equal waves, whose convexities shall lie alter-

nately on opposite sides of the axis (26). Now DB must be the first wave, because the impulse is made on *o* its middle point, and the extremity B is immoveable; therefore as DB makes an effort to excite an undulation in AD, it will effect it by repeated essays, if AD be capable of such a motion. But AD, being double of DB, is susceptible of such a motion; consequently this undulatory motion will be regulary propagated from DB to DC, and from DC to DA, each wave vibrating in the same time, and the convexities lying successively on opposite sides of the axis. And because each vibrating part is one-third of the whole string, the note will be the twelfth major above the fundamental. If the bow be drawn gently along at *m* or *n*, the effect will be the same; because DB will be necessarily set in motion, and therefore communicate its motion as before.

58. To prove that the motion by which harmonic tones are produced, is merely undulatory, draw the bow lightly along the middle of oB , and the harmonic note will be the octave of the former; that is, the nineteenth above the base note. For the breadth of the first wave is oB (27), and it will endeavour to generate a series of *equal* waves, whose convexities shall lie successively on opposite sides of the axis (26). Now the string is susceptible of such motions, because oB is an aliquot part of the whole string (27), *viz.* one-sixth, and the obstacle, pressing at D , lies at the intersection of two of the waves, and therefore does not interrupt the series. Consequently the harmonic note, being produced by one-sixth of the whole string, will be the octave of the former note, that is, it will be the nineteenth above the fundamental. We may observe in general that slight pressures on any or all the nodes, or intersections

of the waves, cannot disturb the undulation, for the reason just now assigned.

This experiment, where there is but one slight pressure at D, and the bow drawn along the middle of oB to produce the nineteenth, requires that the string should be of a considerable length; otherwise the entire portion DB will be set in motion, and produce its proper tone.

59. A beautiful experiment mentioned by Bernouilli will confirm this reasoning. He contrived a wheel with teeth which struck against a musical string, so as that it should receive two impulses in the time of one vibration of the entire string. The effect of these repeated strokes was, that the string visibly resolved itself into two equal convexities lying on opposite sides of the axis. For the impulses endeavour-

ed to generate a wave which should vibrate in half the time of the vibration of the whole string; and this wave endeavoured to produce another contiguous to it whose convexity should always lie on the opposite side of the axis (26). Therefore as the string was susceptible of such a motion, it was at length impressed by the repeated efforts of the wheel.

But without any pressure whatsoever being made on any part of the string, we can often, on long strings, produce these acute harmonics. For according to the force with which the bow is moved, and the point where it is drawn along, the first waves excited by the impulses will be of different latitudes, which producing their proper series of equal waves will generate different harmonic notes. Thus by drawing a bow very lightly and rapidly towards the end of the string of a violoncello, we fre-

quently produce the octave, twelfth, fifteenth and seventeenth. But any of these harmonics will be more certainly generated in this manner, without any slight pressure on the string, if the pressure be first made, and the harmonic note produced in the usual way; and if, immediately after, the bow be lightly drawn along, near the middle of the extreme aliquot parts; for the motion, generated by means of the pressure, still subsisting in the string in some small degree, will be easily renewed by the impulse on the middle of either of the extreme waves. The motion in these cases is evidently undulatory, as it is communicated from one extremity of the string to the other.

60. Diderot has accounted for this phenomenon in another manner, (*see his Oeuvres Philosophiques, tom 6, p. 50*); he conjectures, that the first vibrations impressed on a string by an impulse,

are more rapid the more distant the point of impulse is from the middle ; and that it is only after a certain number of vibrations that the string assumes such a form, as that all its points will arrive at the axis at the same instant ; in which case alone, he says, the vibrations become isochronal. But Euler, (*Acta Petrop. V. 17, p. 394*) has demonstrated, that the time of the vibration is the same in all cases, and that it by no means depends on the figure which it assumes in vibrating. Besides it is not only the *part* of the string, but the *manner* in which it is struck, that renders the tone more acute. And were Diderot's hypothesis true, we should as frequently hear discords as concords ; for certainly there is no reason why, in these accidental variations of the velocity, those degrees only should be produced on which concords depend. But we find that no other notes are ever produced in this manner, but such as

appertain to aliquot divisions of the string.

Some authors maintain, that the only reason which can be assigned for the same string returning these different tones, must be the different force of its strokes upon the air. In one case, say they, it has double the tone of the other ; because, upon the soft touches of the bow, only half its elasticity is set in motion. (*See Encyclopæd. Britan. article Acousticks.*) What is meant by setting in motion only half, or one-third, or any other fraction of the elasticity of the string, I confess is unintelligible to me. But certainly harmonic tones may be produced in this manner without pressure, which are much stronger than the dying tones of the entire string. And farther, according to this theory, as on Diderot's, we should hear all manner of discords as frequently as the octave, twelfth, or se-

venteenth ; for there is no reason why these accidental touches of the bow should not happen to set in motion any other fraction of the string's elasticity, whatever may be meant by that expression, as frequently as its aliquot divisions.

61. That the resolution of strings in harmonic tones into equal parts, arises from an undulatory motion, in which the waves must consequently vibrate in the same time, and their convexities lie successively on opposite sides of the axis (26), will farther appear from this experiment ; that if you cause the parts to vibrate in different times, and consequently destroy the undulation, there will be no harmonic tone generated. Thus if the bow be drawn along the node C (*fig. 22*), no harmonic note will be produced ; for the bow then agitates the portion AD, which endeavours to communicate a similar motion

to DB, so as that the waves should successively lie on opposite sides of the axis; but as DB vibrates in half the time of AD, such a motion cannot be impressed, and therefore there is no tone. Again, DB, vibrating by the motion communicated to it under the slight pressure at D, endeavours to propagate a similar series of waves to AD, but as all AD is kept in motion by the friction at C, neither can this subordinate motion be impressed on it. Therefore no regular motion can be generated, the two parts on either side of the pressure counteracting each others effects: consequently there can be no harmonic tone.

62. If the bow be drawn very violently along m or n , the middle points of AC and CD, even in this case there will be no harmonic, for the same reason as before; because the friction is so violent as to agitate the whole string

AD, and not to admit of its resolution into waves ; therefore there will be no musical tone, as in the case where the friction was at C.

63. But if the bow be drawn gently along the middle of any of the aliquot parts, suppose at m or n , so as not to agitate violently the entire portion AD, but to admit of the string's resolution into these aliquot divisions, there will be the same harmonic tone produced in all the different cases, the pressure continuing on the same point C.

64. If AD be not a multiple of DB (*fig. 31*), but yet commensurate to it, as for instance, if AD be to DB as 3 to 2, the harmonic tone will be in unison with that portion of the string which is the greatest common measure of the two parts on either side of the pressure ; that is, in this case it will be the octave above DB, or

the seventeenth above the fundamental. M. Sauveur made this experiment. The cause of the effect is this : the friction being made nearly on the middle of Bm , will endeavour to agitate the whole string, and also to generate a series of equal waves, whose latitude shall be equal to Bm (27) : the whole string cannot be set in motion, because of the pressure at D , there remains therefore the undulation only ; and this series of waves will not be interrupted by the light pressure at D , because it is the extremity of one of the waves. Thus will the whole string be resolved into five parts, each causing the convexity of the contiguous wave to lie on the opposite side of the axis ; and as these parts are equal, and therefore their vibrations synchronal, they will not interrupt each other's motions, and consequently this undulation will be continued undisturbed ; therefore the harmonic will be in unison with the tone

of one of these five equal parts, or the octave to DB ; that is, the seventeenth above the fundamental.

Even if the entire portion DB were at first set in motion, but that agitation not violent, yet the motion must terminate in such an undulation as we have described : for DB being set in motion, would generate another wave D_0 equal and contiguous to it ; but D_0 could produce no other equal wave ; A_0 would then be agitated, and would endeavour to generate a series of recurring waves (27), whose common latitude would be A_0 ; and this latitude being an aliquot part of the whole, this motion would take place in the string.

65. If the bow be drawn along the node m , there will be no harmonic ; for the reason mentioned in (61).

66. The same harmonic tone may be produced by pressing the string in several different parts; provided the greatest common measure of the two parts of the string on either side of the pressure be the same in these different cases (64). Thus in the preceding experiment the harmonic will be the same whether the pressure be at m , D , n , or o .

67. That the cause of harmonic tones is the synchronism in the vibration of the portions of the string, by which means the undulation is continued undisturbed, the convexities of the waves lying successively on contrary sides of the axis; and that they do not depend on any property in the string of resolving itself into equal parts, will appear from the following experiment: Let the string AB (*fig. 23*) be not of the same diameter throughout, but the part AD somewhat thicker than DB ; and let the light pressure be near D , sup-

pose at d , so as that the tone of dB shall be the octave above that of Ad ; the string will in this case be resolved into three *unequal* unisonal parts. For the wave dB endeavours to produce another in the contiguous portion dC , whose convexity shall always lie on the opposite side of the axis; and this portion dC will endeavour to effect the like in AC . Therefore if the string be capable of having such a motion excited in it, it will be effected by the repeated efforts of dB ; but as dB is the acute octave to Ad , Ad may be resolved into two unisonal parts with dB ; therefore the three parts AC , Cd , dB will vibrate in the same time; and the string Ad being susceptible of the motion which dB endeavours to excite in it, that motion will in fact be excited.

68. The same mode of reasoning may be extended to those cases, where the ratio of the number of vibrations per-

formed by one part of the unequally thick string, to that of the number of vibrations performed in the same time by the other, is not as unity to any whole number, but in any commensurable ratio ; and the tone produced will be in unison with that of a string which is the greatest common measure of two strings of the same diameter, density and tension, which are in unison with the two parts into which the irregular string is divided.

To confirm this by experiment ; I took the compound string mentioned above, and by a moveable bridge divided it so in F (*fig. 24*), that the portion AF sounded a fifth above the tone of FB ; then removing the bridge, and gently pressing the string at F, and drawing a bow lightly along it, an harmonic tone was produced which was the acute octave to that fifth. Here the portion AF was resolved into two equal

parts; and the portion FB into three unisonal parts, which must have been unequal, because the string FB is composed of two cylinders of different diameters.

69. To carry this experiment farther; I caused the thinner string CB (*fig. 24*) to sound exactly a major third above AC; and lightly pressing the string at C, and drawing the bow gently over it, I heard the acute double octave to the tone of CB. Here the string CB resolved itself into four unisonal parts, and AC into five; for the major third is to its fundamental as 4 to 5, the greatest common measure of which is unity; and one-fourth unisonal part of CB sounds the double octave above it. But the parts of AC must have been shorter than those of CB, because of the greater diameter of AC.

70. Again, instead of placing the bridge at C, I placed it a little nearer to B, as at G, and fixed another bridge firmly at M, so that GB founded an acute octave to the major third above MG; then removing the bridge at G, and pressing the string there lightly, I drew the bow gently over it, and heard the double octave to the major third above MG. Because the acute octave to the major third above MG is to the tone of MG as 2 to 5, the greatest common measure of which is unity; but one-half of GB is the acute octave to GB, and consequently the double octave to the major third above MG. Here GB was resolved into two *equal* parts, and MG into five *unisonal* and consequently *unequal* parts as before.

From this variety of experiments we see, that harmonic tones are produced by an undulation; that the tone produced is to the fundamental as the

greatest common measure of the numbers expressing the proportion of the number of vibrations performed in a given time by the two portions of the string on either side of the pressure, is to their sum ; and that they can only take place when the vibrations of the waves are synchroal. We may observe, that the acute harmonic tones are remarkable for their peculiar softness and sweetness, whence they are sometimes called flute tones, from the similitude they bear to the notes of that instrument. This peculiarity in their quality may be probably accounted for in the following manner : When tones are produced on any stringed instrument by a bow, it acts on the strings, by means of the adhering resin, in the same manner as the teeth of a wheel. We may therefore conclude, that the vibrations of the string must be interrupted and rendered considerably irregular by this friction ; and the more

violent the friction, the greater must be the irregularity and disturbance of the vibrations, and consequently the harsher and more grating must be the tone: which we find by experiment to be the case. Now the harmonic tones, on this principle, should be peculiarly soft and delicate, both because they are generated by a very slight and gentle friction; and also, principally, because, there is but one of the several aliquot portions of the string which produce them, on which any friction at all is exerted. If this be the true solution, we see why wind instruments should produce sweeter and softer tones than stringed instruments, namely, because their tones are generated without friction. And this is not only true in wind instruments, but in stringed instruments also, which produce tones as soft as those of wind instruments, when these tones are generated without friction, as is the case in the harp of Æolus. In the mu-

fical glasses indeed, and in the instrument called the Harmonica, the tones, though remarkable for their softness, are produced by friction; but the surface of the finger is so smooth, that no considerable interruption can take place in the vibrations of the glass; and we may observe, that the tone is of a much finer quality, after the removal of the finger, while the vibrations of the glass continue; but if the surface of the finger be rendered rough, by moistening it with an acid, the tone will be rendered in a considerable degree harsh and grating.

S E C T. V.

OF THE HARP OF ÆOLUS.

71. **T**HE phænomena of the Æolian Lyre may be accounted for on principles analogous to those by which we have attempted to explain the phænomena of sympathetic tones in section 2. This pleasing instrument, which has been reputed by some to be a modern discovery, was in truth the invention of Kircher, who has treated largely of it in his *Phonurgia*, (*see page 148*). It is an instrument so universally known, that it may well be presumed unnecessary to give any account either of its construction, or the manner of using it.

To remove all uncertainty in the order of the notes in the lyre, I took off all the strings but one; and on placing the instrument in a due position, was surpris'd to hear a great variety of notes, and frequently such as were not produced by any aliquot part of the string; often too I heard a chord of two or three notes from this single string. From observing these phænomena, they appeared to me so very complex and extraordinary, that I despaired of being able to account for them on the principle of aliquot parts. However, on a more minute enquiry, they all appeared to flow from it naturally and with ease.

But before we proceed to examine the phænomena, let us consider what will be the effect of a current of air rushing against a stretched elastic fibre. The particles which strike against the middle point of the string will move the whole

string from its rectilinear position, and as no blast continues exactly of the same strength for any considerable time, although it be able to remove the string from its rectilinear position, yet, unless it be too rapid and violent, it will not be able to keep it bent; the fibre will therefore, by its elasticity, return to its former position, and by its acquired velocity pass it on the other side, and so continue to vibrate and excite pulses in the air, which will produce the tone of the entire string. But if the current of air be too strong and rapid, when the string is bent from the rectilinear position, it will not be able to recover it, but will continue bent and bellying like the cordage of a ship in a brisk gale. However though the whole string cannot perform its vibrations, the subordinate aliquot parts may, which will be of different lengths in different cases, according to the rapidity of the blast. Thus

when the velocity of the current of air increases, so as to prevent the vibration of the whole string, those particles which strike against the middle points of the halves of the string, agitate those halves as in the case of sympathetic and secondary tones; and as these halves vibrate in half the time of the whole string, though the blast may be too rapid to admit of the vibration of the whole, yet it can have no more effect in preventing the motion of the halves, than it would have on the whole string were its tension quadruple; for the times of vibrations in strings of different lengths, and agreeing in other circumstances, are directly as the lengths; and in strings differing in tension, and agreeing in other circumstances, inversely as the square roots of the tensions (*see Smith or Malcolm*): and therefore their vibrations may become strong enough to excite such pulses as will af-

fect the drum of the ear: and the like may be said of other aliquot divisions of the string. In the same manner as standing corn is bent by a blast of wind, and if the wind be sufficiently rapid, it will have repeated its blast before the stem of corn can recover its perpendicular position, and therefore will keep it bent. But if it decays in rapidity or strength, the stem of corn will have time to perform a vibration before it is again impelled; and thus it will appear to wave backwards and forwards by the impulse of the wind. Those particles which strike against such points of the string as are not in the middle of aliquot parts, will interrupt and counteract each others vibrations, as in the case of sympathetic and secondary tones, and therefore will not produce a sensible effect. That we may be more fully perswaded of the truth of these principles, I shall here set down the order of the æolian notes as

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TABLE of Notes produced by a SINGLE string of an ÆOLIAN LYRE



accurately as a good ear could discover.

72. *Observation 1.* The original note of the string being the grave fifteenth to low F on the violin, the æolian notes, as given in the annexed page, were distinctly perceived, and nearly in the same order in which they are set down.

From the table of proportions in Smith's Harmonics, p. 10, we may see, that these notes were produced by such aliquot parts of the string as are denoted by the fractional indexes which are written over them, agreeable to the theory laid down.

73. *Observation 2.* While some of these notes were sounding, I applied an obstacle indifferently to any point which divided the string into such aliquot parts as would produce these notes,

and the æolian note was not interrupted: but if I placed it on any other part, the tone was instantly extinguished. This evidently shews, that the entire string is in fact resolved into such parts, as from the preceding chain of reasoning we should have been induced to prescribe for it.

74. *Observation 3.* I applied an obstacle slightly against the string, so as that its distance from the extremity should be an aliquot part of the whole; and the æolian note was that which would be produced by such an aliquot part: thus we may in general pre-determine what note the harp shall sound. But this effect will not invariably take place; because, though the obstacle may determine the string to resolve itself into such aliquot parts rather than any others, yet the blast may be too strong or too weak, to admit of such a part's vibrating with

sufficient strength to produce a sound ; however, if any note be produced in this case, it must either be that of this very aliquot part, or of some of its own aliquot divisions ; for the obstacle must necessarily determine one of the intersections of the equal indentures.

75. *Observation 4.* When the blast rises or falls, we find the tone also gradually rise or fall : because, as the blast rises, it grows too strong to admit of the vibrations of the longer aliquot parts ; the vibrations of the shorter aliquot parts therefore will predominate, and will gradually shorten as the blast rises in strength. But in cases of sudden variations in the strength of the blast, there will be also sudden transitions in the tones.

76. *Observation 5.* We sometimes hear a chord consisting of two or

three æolian notes; because the blast which is of such a degree of strength as to admit of the vibrations of certain aliquot parts, may also admit of the vibrations of other parts, if they be not very different in length; for their vibrations will be performed in times not very different. But if the lengths of these parts, and consequently their times of vibration, be very different, the blast that admits of the vibration of the one will prevent that of the other. Accordingly, in looking over the foregoing table, we find that the chords consist of those notes which are produced by such different aliquot parts as are least unequal: thus, one chord consists of C and E, which notes are produced by one-sixth and one-seventh of the string. Another chord consists of F and A, which are produced by one-fourth and one-fifth of the string. Another consists of A, C and E, which notes are produced by

one-fifth, one-sixth and one-seventh parts of the string.

It is also worthy of observation, that in long strings we never hear the original note and its octave at the same time ; because though they are the next aliquot parts, yet their difference is so great, that the blast which admits of the vibration of one of them, will obstruct and prevent the other. It is only in the higher divisions of the string that the chords are heard at all ; and the sharper the note, the more frequent are the chords, for the reasons assigned above, namely, because the different aliquot parts, in such cases, approach nearer to equality.

77. *Observation 6.* Æolian tones are often heard, which are not produced by any exact submultiple of the string : but such notes are very transi-

tory, and immediately vary their pitch, gradually falling or rising to the notes next below or above them, which are produced by exact aliquot parts of the whole string. This arises from the transition of the divisions of the string from one number to another; for during this transition, the parts of string, whose vibrations produce the note, are gradually lengthening or shortening. Thus, suppose the æolian tone was produced by one-third of the string AB, and that the breeze so varies as to cause this tone to fall into the octave of the original note; the points C and D (*fig. 22*), will gradually run towards *n*, and by so doing, will produce a note gradually flattening until it terminates in the octave to AB.

78. Discords are also often heard from the unison strings of this instrument; the cause of this is also evident from the manner in which the notes are ge-

nerated. For the aliquot parts of a string contain in themselves an infinite variety of discords. See page 92.

79. Kircher, in his *Phonurgia*, page 148, has attempted to account for these phenomena of the æolian lyre, by supposing the current of air to strike on different portions of the string. But this is absolutely overturned by experience; for suppose the æolian note to be one-fifth above the original note of the string; that is, produced by AC, one-third of the whole AB (*fig. 22*); then, according to Kircher, the remaining part CB would be at rest, which is false; for an obstacle applied to any other point than C or D will destroy the æolian tone. Besides, the chords that would arise on this theory, are not such as really take place in nature: thus, where the chord consists of the notes F and A (*see the plate for the æolian lyre*), the first note F is produced, according

to Kircher, by the blast's striking on one-fourth of the string ; now in this case the remaining part of the string must be at rest, according to Kircher, but contrary to experience ; or if it be agitated as one firing, it must produce the note of three-fourths of the whole string ; that is, a fourth above the base note ; whereas the note really produced is the double octave to the third above the base note, as may be seen in the table of the æolian tones.

S E C T. VI.

OF GRAVE HARMONIC
TONES.

80. **B**ESIDES the harmonic tones treated of in sect. 4, there is another species of them which may be called *grave harmonics*, because they are graver than the principal tones which generate them. These tones were discovered by Sig^r. Tartini. However, M. D'Alembert informs us, that M. Romieu had made the same discovery a short time before him. Tartini, who founded his system of harmony on these grave harmonics, in the same manner as Rameau derived his from the acute, has given an account of his experiments, in his *Trattato di Musica secondo la vera scienza*

dell' armonia, printed at Padua, 1754. The original work I have not seen, but have taken the following extract respecting it from the *Encyclopædie*, article *Fondamental*; and from *Roussseau's Musical Dictionary*, article *Système*.

81. "Two sounds (says Tartini) being produced at the same instant, by an instrument capable of holding the note, that is, continuing it for some time without intermission, as a trumpet, hautbois, violin, &c. these sounds will produce a third sound sufficiently sensible. Thus, if on the same violin, and at the same instant, you produce two strong and continued sounds, in any relation whatsoever to each other, they will produce a third sound, which will be different in different intervals of the principals. The same thing will happen if, instead of producing these two sounds from the same violin, you produce them separately,

but at the same instant, from two violins distant from each other about five or six paces. The hearer being situated between the instruments will perceive the third sound ; and that the more distinctly, the nearer he is to the middle point between them. This experiment succeeds better with hautbois or wind-instruments, than with violins.

The third sound produced in different cases is as follows :

Two sounds at the interval of a fifth, generate the unison of the grave generator.

A fourth minor generates the lower octave to the acute, or the fifth to the grave generator.

A third major, the lower octave to the grave.

A sixth minor, the double octave to the acute.

A third minor, the tenth major of the grave.

A sixth major, the tenth major of the acute.

A tone major, the double octave to the grave.

A tone minor, the double octave to the third major of the acute.

A femitone major, the triple octave of the acute.

A femitone minor, the twenty-sixth of the grave."

He has also observed a singular consequence which follows from these ex-

periments, *viz.* that if the generating sounds be

ut, sol, ut, mi, sol, ut, re, mi, sol, si, ut,
 $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{1}{12}, \frac{1}{15}, \frac{1}{18},$
*sol, sol, *, &c.*
 $\frac{1}{24}, \frac{1}{27},$

the third note resulting from any two consecutive notes in this progression will be the same, *viz.* the lowest $\frac{ut}{2}$ in the series. He also observes, that the unison and octave are the only intervals which do not produce a third sound.

82. These are the phænomena of Sig^r. Tartini's *grave harmonics*: however the third sound produced in all these different cases should be depressed an octave, as was determined at a meeting of several accurate judges, before whom these experiments were carefully repeated. And this correction is conformable to the account

which M. Romieu presented to the Academy of Sciences at Montpellier, in which he informs us, that he found the third tone to be that of a sounding body, whose number of vibrations was the greatest common measure of the numbers denoting the synchro-
nal vibrations of the generators ; that is, an octave lower than according to Tartini. The physical solution of these phenomena shall be now proposed.

83. The *acute harmonics* are derived from some submultiple resolution of the sounding body, as we have seen in a former section. But the *grave harmonics* cannot by any means be accounted for by considering the sounding body itself; as either a total or partial vibration thereof would produce either a unison, or an acuter note, and therefore cannot generate this species of harmonics, which are graver than the principal sounds. We must therefore have re-

course to the aerial pulses, as the only remaining cause which can contribute towards producing these effects.

84. Two notes of any interval being founded together, it is evident, that if one of the two extreme vibrations of each of the principals coincide, the remaining vibrations will not. Thus, for example, if the interval be a major third, the number of vibrations performed in the same time, by the acute and the grave, will be respectively five and four. Now if the first vibration of the grave coincides with the first of the acute, the fifth vibration of the grave will coincide with the sixth of the acute; because the one performs four vibrations while the other performs five. Thus if the equal portions of time, in which the generators perform their proper number of vibrations, be represented by the equal right lines AB, BC, CD, &c. and MN, NO, OR, &c. (*fig. 30*),

and these lines be subdivided into five and four equal parts respectively, the middle points a, b, c, d, e , of these subdivisions, and m, n, o, p will denote the moments of time in which the pulses are successively produced by these vibrations. Now since mr is equal to MN ; that is, to AB ; that is, to af ; if m be coincident with a , r will be coincident with f : and none of the intermediate pulses will coincide; for if p , for instance, were coincident with e , then mp , which is equal to MP , would be equal to ae ; that is, to AE ; and therefore the number of synchronal vibrations, performed by the generators, would not be as four to five. That is, if the first pulse of the grave generator be coincident with the first of the acute, the fifth pulse of the grave will be coincident with the sixth of the acute; but the intermediate pulses will strike separately on the ear. Now at those instants in which the ear is struck by the

coincident pulses, it will be more affected than at any other. And therefore, besides the two strong sensations produced by the pulses propagated by the vibrations of the sounding bodies, there will be another additional sensation of one impulse on the ear stronger than any others, while the principals respectively perform the number of vibrations denoted by the terms of their ratio. And therefore the effect will be precisely the same as if the coincidences did not increase the intensity of the extreme vibration, but that this effect was produced by any other means; as for instance, by the feeble vibration of a third body, which should perform exactly one vibration while the principals performed their proper number; for every vibration of this body would coincide with one of the extreme vibrations of the principals, and consequently increase their intensity in the same manner as the coincidences. The third

found therefore generated in all cases, should be that of a body which would vibrate once, while the principals performed their proper vibrations.

85. Let us examine now, whether the found, which would arise according to this theory, is such as is in reality produced, conformably to M. Romieu's experiments and our own.

In the interval of a fifth, the number of vibrations performed in the same time by the acute and grave are three and two ; the third found therefore in this case, and in all others also except the octave, being such as would be produced by a body vibrating once in the same time, should be an octave to the grave principal, one to two being the ratio of octaves : which is confirmed by experiment.

In a fourth minor, the vibrations of the principals are four and three. The third found therefore should be such as would be produced by one vibration in the same time that the principals perform four and three; therefore it should be the double lower octave to the acute, one to four being the ratio of a double octave; or it should be the twelfth to the grave, one to three being the ratio of a twelfth: which is also confirmed by experiment.

In a third major, the vibrations of the principals are five and four; the third found therefore should be the double lower octave to the grave, one to four being the ratio of a double octave: which is confirmed by experiment.

In a sixth minor, the principal vibrations are eight and five; the third found should be therefore the triple oc-

tave to the acute principal, one to eight being compounded of the ratios one to two, two to four, and four to eight; that is, three octaves.

In a third minor, the principal vibrations are six and five. Now one to five is compounded of the ratios of one to two, two to four, and four to five; that is, of two octaves and a third major, that is, a seventeenth. Therefore the third sound should be a seventeenth major to the grave principal: which is conformable to experience.

In a sixth major, the principal vibrations are five and three; but one to five is a seventeenth major as before. Therefore the third sound should be a seventeenth major to the acute principal: which is conformable to experience.

In a tone major, the principal vibrations are nine and eight; but one to

eight is compounded of the ratios one to two, two to four, and four to eight, that is, three octaves. Therefore the third found should be a triple octave to the grave principal: which is confirmed by experience.

In a tone minor, the principal vibrations are ten and nine; but one to ten is compounded of the ratios one to four, four to five, and five to ten, that is, of a double octave, a major third and an octave. Therefore the third found should be the triple octave to the major third of the acute principal: which is conformable to experience.

In a major semitone, the principal vibrations are sixteen and fifteen; but one to sixteen is compounded of the ratios one to two, two to four, four to eight, and eight to sixteen, that is, four octaves. Therefore the third found should

be the quadruple octave to the acute principal : which is conformable to experience.

In a minor semitone, the principal vibrations are twenty-five and twenty-four ; but one to twenty-four is compounded of the ratios one to eight, that is, a triple octave ; and of eight to twenty-four, or one to three ; that is, an octave to the fifth, or a twelfth. Therefore the third sound being composed of a triple octave and a twelfth, should be a thirty-third to the grave principal : which is conformable to experience.

86. Tartini observed, as I mentioned above, that the intervals of the unison and octave produce no third sounds ; the reason of which is evident from the preceding theory. For in the unison, every successive vibration of the one sounding body coincides with every

ſucceſſive vibration of the other ; conſequently all the ſucceſſive impulſes on the ear are of the ſame magnitude ; and therefore there cannot be any third ſound generated. In the octave, every vibration of the grave coincides with every alternate vibration of the acute ; and therefore every alternate impulſe on the ear will be ſtronger than that immediately preceding it. Now this effect would tend to generate the ſenſation of an octave below the acute, or the unison of the grave ; which effect will therefore be inſenſible, it being very difficult to diſtinguiſh octaves or uniſons when ſounding together ; but particularly in this caſe, where the tone is ſo exceedingly feeble.

The third ſound reſulting from the interval of the minor fourth, and from the two thirds, and two fixths, whether major or minor, is the moſt eaſily diſtinguiſhed by the ear ; becauſe there

is always one of the principals to which it is not an octave, and it is not in unison with either of them. But the third found resulting from a fifth, is perceived with difficulty, because it is an octave to the grave principal. And it is distinguished with more difficulty in the major and minor tones, because as they differ but little from each other, the intonation easily confounds them; and the difficulty in the major and minor semitones is still greater, their difference being still less.

87. If three principal notes were sounded together, equally strong and even, a very delicate ear would distinguish three of these grave harmonics at the same time; which are produced according to the law assigned above, from the united effect of each of the two extreme principal sounds with the intermediate, and with each other. Some of these grave harmonics would, in some cases,

be the same ; in others they would be different. Thus, if the synchronal vibrations performed by the three bodies were two, three, and six, the combination of the first and second would produce the octave below the first ; the combination of the second and third would produce the unison of the second ; and the combination of the first and third would produce the same third sound as the combination of the first and second. And were the human ear still more delicate, it would perceive distinct sounds arising from the coincidences of the pulses of these secondary sounds with those of the principals and of each other, and so on *ad infinitum*.

88. It has been very acutely objected to the doctrine of coincidences, that if the first pulses propagated from the sounding bodies do not arrive at the ear at the same instant, they will in some cases never coincide. This we must ac-

knowledge is true ; but then there will in such cases be always two pulses which will very nearly approach to an actual coincidence, and as the effects of all impressions on the organs of sense are not instantaneous, but continue for some time, these pulses, when they make their nearest approach to each other, will have the same effect as if they had actually coincided : and after they have made this their nearest possible approach, they will continue afterwards to preserve that interval, because the two bodies perform their proper vibrations in the same time, and consequently the same impression will return as regularly as if they had actually coincided.

89. Tartini observed, as is mentioned above, that any two consecutive sounds in the progression, *ut* ($\frac{1}{2}$), *sol* ($\frac{1}{3}$), *ut* ($\frac{1}{4}$), &c. would always produce the same third sound, *viz.* the octave to *ut* ($\frac{1}{2}$).

the lowest in the series. The reason of this is evident from the principles now laid down; for the number of simultaneous vibrations of these notes are denoted by the denominators of the fractions of this series: suppose then the two principals founded together are *mi* ($\frac{1}{10}$) and *sol* ($\frac{1}{12}$); while *sol* ($\frac{1}{12}$) vibrates twelve times, *mi* ($\frac{1}{10}$) vibrates ten times; but the third found arising from *mi* and *sol* is such as would be produced by a body vibrating once, while *mi* vibrates ten times; and therefore will be the octave below *ut* ($\frac{1}{2}$); and the same argument holds in all other cases.

90. M. D'Alembert invited the musicians to determine what third sound would arise from the interval of a triton, and a false fifth. M. Romieu answered, from experiment, that the third sound resulting from the latter interval, was the quintuple octave to the acutest of

the principals; contrary to his own law of the grave harmonics. For the ratio of a false fifth is sixty-four to forty-five; and unity, as before, denotes the simultaneous vibration of the sounding body which would produce the third found. Now the ratio of two to sixty-four is compounded of the following ratios, one to two, two to four, four to eight, eight to sixteen, sixteen to thirty-two, and thirty-two to sixty-four, that is, six octaves. But in such delicate sounds as these, it is a very difficult matter to distinguish a note and its octave, which are often confounded in cases where they are much more sensible than in these grave harmonics. Romieu has not determined what third found results from a triton or redundant fourth: according to the preceding theory it must be a quintuple octave to the grave principal; for the ratio of this interval is forty-five to thir-

ty-two ; but the ratio of one to thirty-two is compounded of the following ratios, one to two, two to four, four to eight, eight to sixteen, and sixteen to thirty-two, that is, five octaves.

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 eight, eight to sixteen, and sixteen to
 thirty-two, that is five octaves.

